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The Undercut Procedure

An Algorithm for the Envy-Free Division of Indivisible Items

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Abstract We propose a procedure for dividing a set of indivisible items between two players. We assume that each player's preference over subsets of items is consistent with a strict ranking of the items, and that neither player has information about the other's preferences. Our procedure ensures an envy-free division—each player receives a subset of items that it values more than the other player's complementary subset—given that an envy-free division of "contested items," which the players would choose at the same time, is possible. We show that the possibility of one player's undercutting the other's proposal, and implementing the reduced subset for himself or herself, makes the proposer "reasonable," and generally leads to an envy-free division, even when the players rank items exactly the same. Although the undercut procedure is manipulable and its envy-free allocation may be Pareto-inferior, each player's maximin strategy is to be truthful. Applications of the procedure are discussed briefly.

Keywords: fair division, allocation of indivisible items, envy-freeness, ultimatum game

1 Introduction

In this paper we propose a procedure for dividing a set of indivisible items between two players, who are assumed to have preferences over all subsets of items such that no two individual items are equally preferred. Our procedure ensures an *envy-free* division—each player receives a subset of items that it values at least as much as the other player's complementary subset—provided that there is an envy-free division of "contested items," which are items the players would choose at the same time (timing will be discussed later). Such a split is envy-free, because neither player will envy the other player if each obtains a portion it values at least as much as the other player's portion.[1]

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[1] The concept of "envy-freeness" goes back at least to Foley [8]; for an overview of its use in the fair-division literature, see Brams [3] or Klamler [9]. Most of the literature we cite in this article relates to algorithmic fair division; there is also an extensive literature on axiomatic fair division, which focuses on characterizing different kinds of fair division—what

Remarkably, such a split is always possible—even if the players rank the items exactly the same—as long as they differ on a "minimal bundle" of items. A *minimal bundle* for a player is a subset that is worth at least as much as its complement, but it would no longer have this property if any item in the subset were replaced by a lower-ranked item, or eliminated.

The procedure we describe is a discrete analogue of the well-known "I cut, you choose" procedure for cake-cutting. However, instead of A's giving B a choice between two portions that A values equally, which may be impossible with indivisible items, A, whom we assume is male, proposes a division so he will not envy B's portion. Player A is thus motivated not to ask "too much" for himself, because B, whom we assume is female, has the option of *undercutting* A's portion by taking a slightly less valuable subset for herself. Thereby the players will have an incentive to propose subsets for themselves that are preferable yet, in a sense, minimal. In particular, if a player has no information about the preferences of an opponent, he or she will propose some minimal bundle.

The paper proceeds as follows. In section 2, we introduce the formal framework and give the assumptions and rules of our algorithm, called the *undercut procedure* (UP), which we will illustrate. In section 3, we give a method for identifying all possible minimal bundles of the contested items, enumerating them for up to 7 items. The number of these bundles increases rapidly with the number of contested items, making it increasingly likely that A and B will not have the same minimal bundles and, therefore, that there exists an envy-free split. We also show in this case that, if the players are *sincere* (i.e., report their rankings truthfully), and their preferences over subsets of the items are "responsive" (to be defined), then UP guarantees an envy-free division of the items.

It turns out that sincerity is also a maximin strategy. It maximizes the minimum a player can guarantee for himself or herself, regardless of the strategy of the opponent, given the players have no information about each other's preferences. This result will be extended in the Appendix, where we show that sincere rankings are not necessarily optimal if the players have complete information about each other's preferences. Because information is incomplete in most real-life situations, risk-averse players are well-advised to choose sincerely.

In section 4, we calculate the expected number of contested items when all orderings are equiprobable. As the number of items increases, the expected number of contested items also increases without bound, but only very slowly, so the *proportion* of contested items drops rapidly. Because larger sets of contested items make it more likely that players' minimal bundles will not be identical, the probability that UP will produce an envy-free allocation increases.

If the preferences of the players are positively correlated—as they might well be—then a larger proportion of the items will be contested. But even if all items are contested because the players rank them exactly the same, it is still likely that there will be an envy-free division if there are more than about five or six items. Finally, section 5 concludes the paper.

2 The Undercut Procedure

To preview UP, consider the simpler problem of dividing a single divisible good, such as money. In the *ultimatum game*, for example, one player (A) proposes a share of \$1 to the other player (B). B may accept A's offer or reject it; if the latter, both players end up with nothing. Experimentally, inequitable proposals by A (e.g., 60 cents for himself) are often rejected by B on grounds of fairness, so both players receive zero.

But now give B the option of undercutting A's proposal by 1 cent and implementing the resulting division. It is not hard to see that only by asking for exactly 50 or 51 cents can A guarantee himself a payoff of at least 50 cents:

- If A demands a larger amount (say, 60 cents) for himself, leaving 40 cents for B, B will undercut A's proposal by 1 cent and obtain 59 cents for herself, leaving A with only 41 cents.

necessary and sufficient conditions must obtain to ensure, for example, the existence of an envy-free, Pareto-optimal allocation of items—but this literature does not say how to obtain it.

– If A offers B more (say, 60 cents) and keeps the remainder (40 cents) for himself, B will accept A's proposal and obtain 60 cents for herself, leaving A with only 40 cents.

By this reasoning, it is in A's interest to offer B exactly 50 cents, and for B to accept, thereby implementing the egalitarian outcome of 50 cents to each player. Alternatively, A could offer a 51/49 split, which B would undercut, again leaving A with exactly 50 cents. These strategies are subgame-perfect Nash equilibria, because neither player can do better by departing from his or her strategy at the offer stage (for A) or the accept/undercut stage (for B).[2]

To return to the case of indivisible items, we assume the following formal framework:

2.1 **Formal Framework**

We consider two players, A and B, who must divide a set of $n > 0$ indivisible items, X. The set of all subsets of X is denoted by $\mathcal{X}$. Each player is assumed to have a strict ranking $P_i \subset X \times X$ over X, i.e., is able to rank the items from best to worst. From player i's preferences P_i on X, we will draw conclusions about his or her preferences on $\mathcal{X}$, denoted by $\succsim_i$. Those preferences satisfy the following condition:[3]

Definition 1 Preference $\succsim$ on $\mathcal{X}$ satisfies *responsiveness* if for all $S \in \mathcal{X}$ and all $x \in S$ and $y \in X \backslash S$, $[S \succ S\backslash\{x\} \cup \{y\} \Leftrightarrow xPy]$, $[S\backslash\{x\} \cup \{y\} \succ S \Leftrightarrow yPx]$, and $S \succ S\backslash\{x\}$.

Thus, any set becomes less preferred whenever an item is removed from it, or replaced by an item that is ranked lower according to the preference P on X. The following definition establishes a relation between any two subsets of X:

Definition 2 Let $S, T \subseteq X$, $S \neq T$, and let P be a strict ranking on X. T is said to be *ordinally less* than S, denoted by $T \leq_{OL} S$, if there exists an injective function $\sigma_{T,S} : T\backslash S \to S\backslash T$ such that for all $x \in T\backslash S$, $\sigma_{T,S}(x)Px$.[4]

Thus, T is ordinally less than S if either T is a proper subset of S, or if T can be obtained from S or a proper subset of S by replacing items originally in S by equally many lower-ranked items.

If we denote the complement of S by $-S \equiv X\backslash S$, then we can define a property that characterizes sets that minimally exceed half of the total value.

Definition 3 Player i regards a set $S \subseteq X$ as *worth at least 50 percent* if $S \succsim_i -S$.

Definition 4 Subset $S \subseteq X$ is a *minimal bundle* (from X) for player i if $S \succsim_i -S$ and, for any $T \leq_{OL} S$, $-T \succ_i T$.

Therefore, a minimal bundle for i is a set S worth at least 50 percent to i, with the special property that any subset of S is worth less than 50 percent, as is any subset of S obtained by replacing some items in S by less preferred items not in S.

We are interested in dividing the items in X between players A and B. Let $S \subset X$; then we say that $(S, -S)$ denotes a *split* of X into subset S for player A and subset $-S$ for player B. The most important idea of fairness is defined next:

Definition 5 For any $S \subseteq X$, the split $(S, -S)$ of X is *envy-free* if $S \succsim_A -S$ and $-S \succsim_B S$.

To distinguish "ties," we make the following definition:

Definition 6 An envy-free split of X, $(S, -S)$, is *trivial* if $S \sim_A -S$ and $-S \sim_B S$.

Thus, in a trivial split $(S, -S)$, each player is indifferent between S and $-S$. We next specify the rules of a procedure that can be used to divide a set of indivisible items.

[2] We assume the game is discrete (e.g., all amounts must be integral multiples of 1 cent).

[3] This and other conditions concerning the ranking of sets of objects are discussed in Barbera, Bossert and Pattanaik [1].

[4] The relation "ordinally less" is the transitive closure of the inclusion relation and the right shift (see Taylor and Zwicker [10], p. 93).

2.2 The Undercut Procedure (UP)

Under the rules that follow, we assume the presence of a referee. But the role of the referee could be played by a computer program, because no human judgment is required—the referee either compares subsets or makes random choices.

1. Players A and B each independently names his or her top-ranked alternative. If they name different items, each player receives the item he or she names. If they name the same item, it goes into the *contested pile*, denoted by $I_c \subseteq X$.[5]
2. This process continues until all items have been named by at least one player.
3. If the contested pile is empty, the procedure ends. Otherwise, each player i identifies his or her set of minimal bundles (denoted by MB_i) from the contested pile and gives this information, in secret, to the referee.
4. If the sets of minimal bundles are different, i.e., $MB_A \neq MB_B$, each player i provides to the referee, in secret, a ranking $\succsim_i$ of his or her minimal bundles. A player (say, A) is chosen at random, and A's top-ranked minimal bundle is considered. If it is not also in MB_B, then it becomes the *proposal*, and A is the *proposer*. If A's top-ranked minimal bundle belongs to MB_B, then B's top-ranked minimal bundle is considered. If it does not belong to MB_A, then it becomes the proposal, and B the proposer. If it does belong to MB_A, then the process continues until a minimal bundle of one player is found that is not a minimal bundle of the other. Then proceed to step 6.
5. If the sets of minimal bundles are the same, i.e., $MB_A = MB_B$, and there exists an S such that $S \in MB_A$ and $-S \in MB_A$ (and, therefore $S, -S \in MB_B$ also), then S becomes the proposal. If there is no minimal bundle S such that $-S$ is also a minimal bundle, then a minimal bundle is chosen randomly and becomes the proposal.
6. Assume that S is the proposal and the proposer is A. Then B may respond by
   (a) accepting the split $(S, -S)$ of I_c (which she should do if $-S$ is worth at least 50 percent to her) or
   (b) undercutting A's proposal, i.e., taking for herself her most preferred subset T of those that are ordinally less than S, making $(-T, T)$ the split of I_c.

   The procedure ends. A player's subset of X consists of all items received in steps 1 and 2, plus the player's share of the contested pile determined in step 6.

Observation 1. If A and B are sincere, they each prefer their items assigned according to rules 1 and 2 to the items assigned to the other player. Thus, augmenting these items with an envy-free split of the contested pile guarantees that each player prefers his or her subset to the other's, and therefore an envy-free split of *all* items (as will be shown later). The crucial step is the division of the contested pile, wherein players have exactly the same rankings. This is because, as reporting proceeds from their most-preferred to their least-preferred items, the item they contest first will be ranked highest in the contested pile, the item they contest second will be ranked second-highest, and so on.

Observation 2. Items in the contested pile may not have been ranked the same *initially*. For example, assume that A ranks four items $1P_A2P_A3P_A4$, and B ranks them $2P_B3P_B4P_B1$. After A gets item 1 and B gets item 2 by rule 1, items 3 and 4 are contested. Although they are ranked 1^{st} and 2^{nd}, respectively, in the contested pile, A ranked them 3^{rd} and 4^{th} initially, whereas B ranked them 2^{nd} and 3^{rd}. While both players get their most-preferred items (1 for A and 2 for B), there is no envy-free division of the two items (3 and 4) in the contested pile.

To illustrate the application of UP, consider the following example:

Example 1 Let the contested pile $I_c = \{1, 2, 3, 4, 5\}$ contain 5 items, and suppose that both $i = A$ and $i = B$ rank them $1P_i2P_i3P_i4P_i5$ from best to worst. Assume that the set $\{1, 2\}$ (which for convenience

[5] This terminology is used for *balanced alternation* in Brams and Taylor [7], ch. 3, who call rule 1 the "query step," although they offer no algorithm for dividing the contested pile. While Brams and Fishburn [4] suggest a series of steps that can be used to determine a "middle" envy-free allocation—one that does not favor one player or the other—we indicate later why UP is a simpler and more practicable procedure.

will be written 12) is A's top-ranked minimal bundle (from I_c), that 12 is not a minimal bundle for B, and that 2345 is a minimal bundle for B. In addition, assume A is the randomly chosen proposer, so his initial proposal is 12 (rule 4).

If B thinks the complement of A's proposal of 12, namely 345, is worth at least 50 percent, then B will accept A's proposal, producing a split 12|345 (i.e., 12 to A and 345 to B). This split is envy-free, because both players believe their portion is worth at least 50 percent. But the fact that B has 2345 as her minimal bundle means that 345, which is ordinally less than 2345, must be worth less than 50 percent to B. Thus, B's only option is to undercut A's proposal of 12. B can propose 13 for herself, which is the most valuable subset that is ordinally less than 12. Because 13 is worth less than 50 percent to A (because 12 was a minimal bundle for him), he must think the complement of 13, namely 245, is worth more than 50 percent.

Thereby both players can obtain at least 50 percent if 13 is worth at least 50 percent to B. How do we know this is the case? Because 2345 is a minimal bundle for B, 245 must be worth less than 50 percent to B. Therefore, the complement, 13, must be worth more than 50 percent.

In general, either the complement of A's proposal, 345 in this case, or some subset that is ordinally less than A's proposal, 13 in this case, must be worth more than 50 percent to B.

This example illustrates that, if rule 4 applies, UP produces a nontrivial envy-free split. Should the players have the same set of minimal bundles, then rule 5 applies. Then we might have a trivial envy-free split, or there may be none. The latter phenomenon occurs, for example, if there is one item that is highly valued by both players, and constitutes the only minimal bundle for both (such as the house in a divorce, which each party may prefer to the totality of all other items). Although UP will lead to ex-ante envy-freeness, there is not ex-post envy-freeness.

The following theorem precludes such situations by assuming that the two sets of minimal bundles are not identical.

Theorem 1 *There is a nontrivial envy-free split of the contested pile if and only if one player has a minimal bundle that is not a minimal bundle of the other player. If so, then UP implements an envy-free split.*

Proof Assume that a nontrivial envy-free split $(S, -S)$ exists. Then there must be a player, say A, such that $S \succ_A -S$. Now, suppose first that $S \in MB_B$, i.e., S is a minimal bundle for B. Hence, $S \succsim_B -S$. But as $(S, -S)$ is an envy-free split, also $-S \succsim_B S$ and, therefore, $S \sim_B -S$. Now assume $T \subset X$ such that $T \leq_{OL} -S$. This implies that $S \leq_{OL} -T$. Given responsive preferences $\succsim_B$, $-T \succ_B S$ and, hence, $-T \succ_B T$. Consequently, $-S$ is a minimal bundle for B.

Now assume a subset S of the contested pile such that $S \in MB_A$ and $S \notin MB_B$. There are two mutually exclusive possibilities:

1. $S \succ_B -S$. Then there exists a $T \leq_{OL} S$ such that $T \succsim_B -T$, i.e., T is worth at least 50 percent to B.
2. $-S \succ_B S$.

Because these cases are mutually exclusive, exactly one applies; in either case, B gets a bundle worth at least 50 percent. At the same time, A also receives at least 50 percent, whether B accepts A's proposal, which is worth at least 50 percent to A, or undercuts his proposal, because the complement of any undercut must be worth at least 50 percent to A. □

Theorem 1 not only gives a condition for the existence of an envy-free split of the contested pile (namely, $MB_A \neq MB_B$), but it also establishes that UP will implement such an envy-free split.

Example 2 Returning to our Example 1, assume now that B is chosen as the proposer, and she proposes her top-ranked minimal bundle, 2345. Now A can either accept the complement, 1, for himself or undercut B by proposing 234. Because 12 is in MB_A, subset 1 is worth less than 50 percent to A. But 15 is also worth less than 50 percent to A, so 234 must be worth at least 50 percent. On the other hand, the complement of 234, 15, must be worth at least 50 percent to B, because 234 is worth less than 50 percent to B. In summary, we have shown that the $A|B$ split is

– 245|13 when A goes first; and
– 234|15 when B goes first,

giving each player at least 50 percent. Moreover, the player who goes second is advantaged—obtaining a more-preferred subset than if he or she went first— at least in the case when this player undercuts the proposer's offer (as here). Because the proposer is selected at random (rule 4), neither player is favored, in expectation, by UP.[6]

As well, it is apparent that there is a "middle" envy-free split in this example, namely 235|14. This allocation gives each player a subset intermediate between the one obtained by going first and by going second.

One problem with algorithms that identify a middle split (Brams and Fishburn [4]), and thereby produce a result that is arguably more balanced, is that they require that the players rank all their envy-free splits—not just their minimal bundles—so that the referee can choose a middle one. But under UP, what might be a middle envy-free split, should B accept A's proposal, may not be a middle one should B undercut it. Thus, we prefer the present rules, which yield *some* envy-free split, as long as $MB_A \neq MB_B$, and treat the players equally in expectation.

Finally, we ask whether a player can ever do better under UP by proposing a subset that is not a minimal bundle. For example, suppose that 1 is a minimal bundle of A, but that A proposes 12 for himself. If 345 is a minimal bundle for B, then she will accept A's proposal and A will obtain 12; 12 is better for him than obtaining only 1 had he proposed just this item for himself, because B would certainly have accepted the complement, 2345. However, if A had proposed 12, B could have undercut, obtaining 13 for herself and leaving 245 for A. The latter subset must be less than 50 percent for him, because even 2345 is less than 50 percent if 1 is in MB_A. Thus, A could do better by misrepresenting his minimal bundle(s), but he also might do worse, obtaining less than 50 percent.

In the division of the contested pile, we say that a strategy is maximin for a player (say, A) if it minimizes the number of ex-ante possibilities in which A receives a share worth less than 50 percent to A. Then we can show:

Corollary 1 *In the division of the contested pile, a player's (say, A's) maximin strategy is to name all his minimal bundles. If $MB_A \neq MB_B$, then A cannot receive less than 50 percent if he proposes a minimal bundle, whereas proposing a nonminimal bundle, or failing to propose all minimal bundles, may lead to his receiving less than 50 percent.*

Proof If $MB_A = MB_B$, then either A obtains exactly 50 percent (if some S and $-S$ are in MB_A), or a proposal will be randomly chosen, in which case A might receive less than 50 percent. Now suppose that $MB_A \neq MB_B$. If A proposes a split in which A receives less than 50 percent, then A's worst-case outcome is for B to accept A's proposal, leaving A with less than 50 percent. If A proposes a split in which the subset A receives is worth at least 50 percent to A and contains a proper subset that is also worth at least 50 percent to A, A's worst outcome is for B to undercut and choose that subset, leaving A with a subset worth at most 50 percent to him. Thus A should propose only splits such that (1) A receives at least 50 percent and (2) any subset is worth less than 50 percent to A. On the other hand, if A proposes a minimal bundle that turns out not to be a minimal bundle of B, Theorem 1 ensures that A will receive at least 50 percent. Finally, not revealing all of his minimal bundles to the referee exposes A to the risk that B proposes one of them, in which case A either must accept its complement (worth at most 50 percent to A) or undercut it (for certain receiving less than 50 percent). Hence, if A truthfully names all his minimal bundles, there is at most one possibility of receiving less than 50 percent, whereas if he fails to do so there is at least one possibility of receiving less than 50 percent; B's situation is similar. □

From Theorem 1 we know that, given $MB_A \neq MB_B$, there is a nontrivial envy-free split of the contested pile. Can we be sure that, given such a split of the contested pile, the division including

[6] There are many examples of games with a first-mover advantage (Chicken, duopoly games) and others, like the game under UP, with a second-mover advantage (more on games and their equilibria in the Appendix).

the items assigned directly via rules 1 and 2 will also be envy-free? Consider the following extension condition for preferences on $\mathcal{X}$:

Definition 7 Preference $\succsim$ on $\mathcal{X}$ satisfies *extension monotonicity* if for all $S, T \in \mathcal{X}$, all preferences P, and all $x, y \in X \backslash (S \cup T)$, $S \succsim T$ and xPy implies $S \cup \{x\} \succ T \cup \{y\}$.

Proposition 1 *Given responsive and extension monotonic preferences of the players and an envy-free division of the contested pile, the final division of X under UP is envy-free.*

Proof Given an envy-free division of the contested pile under UP, we know that a player (say A) at least weakly prefers his share from the contested pile (call it I_c^A) to B's share from the contested pile (I_c^B), i.e. $I_c^A \succsim_A I_c^B$. Moreover, the set of items that A received under rules 1 and 2 of UP, say $X_A \subset X$, is preferred by A to the set of items B received, X_B, given responsiveness. Hence, $X_A \succsim_A X_B$. Because $|X_A| = |X_B|$, we can conclude from repeated use of extension monotonicity that $I_c^A \cup X_A \succsim_A I_c^B \cup X_B$. Because the same reasoning applies to player B, the final split is envy-free. □

3 Feasible Subsets When Players Rank Items Exactly the Same

As we saw, the initial difficulty in applying UP occurs when players name the same item, so it goes into the contested pile, wherein both players rank items exactly the same. We call a subset of contested items *feasible* if a player may strictly prefer it to its complement. A necessary condition for a nontrivial envy-free split is that the subsets assigned to both players be feasible.

Take the set of contested items $I_c = \{1, 2, ..., c\}$, for which A and B have the same ranking P_i on I_c, namely $1P_i2P_i3P_i...P_ic$. If $c = 1$, there is no possible envy-free split, because there is only one item. If $c = 2$, there is also no possible envy-free split, given the strict ranking on I_c and responsive preferences of the players. If $c = 3$, the only possible envy-free split is giving 1 to one player (say, A) and 23 to the other player (B), denoted by 1|23. And if $c = 4$, there are two possible envy-free splits, namely 1|234 and 14|23.[7]

The possible envy-free splits when $c = 5$ are not so obvious. Figure 1 shows all subsets that could possibly be part of an envy-free split. Note that whenever a subset can be part of an envy-free split, so can its complement. In Figure 1, superscripts are used to indicate subsets that are complements, e.g., 145 and 23 in the two middle columns, and 12 and 345 in the left-hand and right-hand columns. Among the subsets shown in Figure 1, some preference relations hold automatically. For example, in the left-hand column, 12 is always preferred to 13, 13 to 14, 14 to 15, and 15 to 1.

In fact, Figure 1 is the Hasse diagram describing all preference relations among these subsets. These relations are indicated by lines (arcs) between nodes (subsets); if there is an upward or upward-sloping path from subset S to subset T, then T is always preferred to S. For example, 12 is always preferred to 15 and to 23, and 145 is always preferred to 1 and 345. But if there is no upward path between two subsets, then their relative preference can vary; for instance, 234 might be more preferred or less preferred than 145, and 15 might be more or less preferred than 245 (or these pairs may be equally preferred).

Feasible subsets were identified in Brams and Fishburn [4] but not characterized. We next give necessary and sufficient conditions for feasibility.

The only assumption put on the players' preferences on the set of all subsets of I_c is the previously defined responsiveness condition. The following gives a rigorous definition of feasibility:

Definition 8 A subset $S \subseteq I_c$ is *feasible* if there exists a responsive preference $\succsim$ such that $S \succ -S$.

For example, if $c = 5$, then both 12 and 345 are feasible; so is 123, but 45 is not, because of our assumption of responsiveness. As we will see, some feasible subsets may be parts of envy-free splits (e.g., 12 and 345), but others (e.g., 123) are not because their complements (45) are not feasible.

[7] Without the assumption of strict rankings of the items, further envy-free splits would be possible, e.g., 1|2 when $c = 2$ or 12|34 when $c = 4$ would be possible envy-free splits in case at least one player values all items identically.

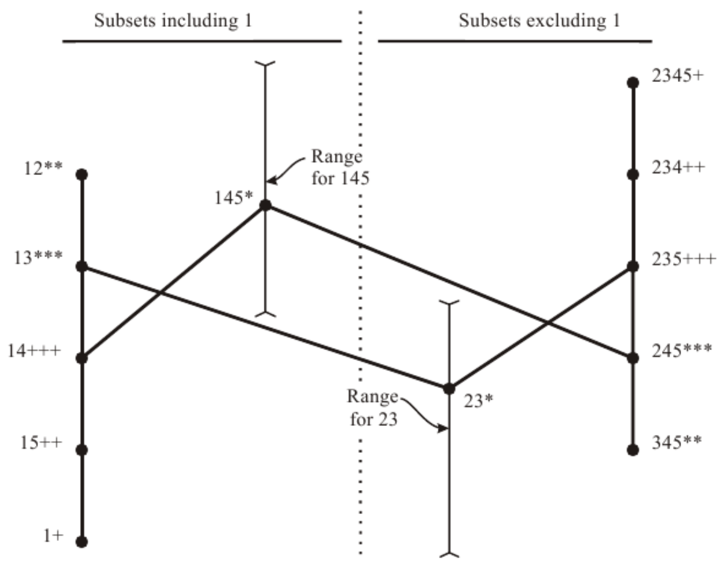

Fig. 1 Feasible subsets S for $c = 5$

The following theorem gives a necessary and sufficient condition for S to be feasible. Recall that, for any positive integer k, $I_k = \{1, 2, ..., k\}$.

Theorem 2 *Let $S \subseteq I_c$ and $S \neq \emptyset$. Then S is feasible if and only if $|I_k \cap S| > \frac{k}{2}$ for some k, $k = 1, 2, ..., c$.*

Proof For S to be feasible requires that there be no injective function $\sigma_S : S \to -S$ such that for all $x \in S$, $\sigma_S(x)Px$; otherwise (by responsiveness of the preferences), $-S \succ S$. There are two cases to be considered:

1. For sufficiency, assume S is feasible. Then, for some m, at most $n < m$ of the m top-ranked elements of S are higher-ranked than n elements of $-S$. But this implies that for some k, $|I_k \cap S| = m$ but $|I_k| < 2m$ and, hence, $|I_k \cap S| > \frac{k}{2}$.
2. For necessity, assume S is not feasible. Then there is an injective function σ_S of the above form. This implies that, for each such S, there must be at least m alternatives in $-S$ that are higher ranked than the first m alternatives of S. Hence, for any k top-ranked alternatives, at most half can be in S, i.e., $|I_k \cap S| \leq \frac{k}{2}$ for all k. □

The condition in Theorem 2 is easy to apply. For example, if there are $c = 4$ contested items and $S = \{2, 3\}$,

- when $k = 1$, the intersection of $\{1\}$ and $\{2, 3\}$ is $\emptyset$, i.e., $|I_1 \cap \{2, 3\}| = 0 \leq \frac{1}{2}$
- when $k = 2$, $|I_2 \cap \{2, 3\}| = |\{2\}| = 1 \leq \frac{2}{2}$
- when $k = 3$, $|I_3 \cap \{2, 3\}| = |\{2, 3\}| = 2 > \frac{3}{2}$
- when $k = 4$, $|I_4 \cap \{2, 3\}| = |\{2, 3\}| = 2 \leq \frac{4}{2}$

When $k = 3$, the number of items in the intersection is $2 > \frac{k}{2} = \frac{3}{2}$, so Theorem 2 shows that $\{2, 3\}$ is feasible.

Theorem 2 has some useful corollaries:

Corollary 2 *For any c, $\{1\}$ and any S containing $\{1\}$ is feasible.*

Proof For any S such that $\{1\} \in S$, and for $k = 1$, $|I_1 \cap S| = |\{1\}| = 1 > \frac{k}{2} = \frac{1}{2}$. □

Recall that for an envy-free split $(S, -S)$, both S and $-S$ must be feasible. We say that $(S, -S)$ is a *possible envy-free split* if there exist two responsive preferences under which both S and $-S$ are feasible.

Corollary 3 *Suppose that $S \subseteq I_c$ and that $\{1\} \notin S$. If S is feasible, then $(S, -S)$ and $(-S, S)$ are possible envy-free splits.*

Proof As $\{1\} \in -S$, by Corollary 2, $-S$ is feasible. It follows that, if S is feasible, then both $(S, -S)$ and $(-S, S)$ are possible envy-free splits. □

It is not hard to show, using Theorem 2, Corollary 2, and Corollary 3, that the complementary subsets given in Figure 1 constitute all possible envy-free splits when $c = 5$. In Table 1, where we extend the analysis to $c = 6$ and $c = 7$; we do not give feasible subsets that include item 1, because they are simply all sets that include item 1 (see Corollary 2).

No. of items in subset	$c = 6$	$c = 7$
2	23	23
3	234,235,236,245,345	234,235,236,237,245,345
4	2345,2346,2356,2456,3456	2345,2346,2347,2356,2357 2367,2456,2457,2467,2567 3456,3457,3467,3567,4567
5	23456	23456,23457,23467,23567,24567,34567
6		234567

Table 1 Feasible subsets S for $c = 6$ and $c = 7$ that do not include item 1

In Table 1, we break down our enumeration of the feasible subsets S according to the number of items in a feasible subset. Note that the possible envy-free splits are exactly the feasible subsets, S, that we enumerate in Table 1 together with their complements, $-S$, which always include 1.

Observe that the modal number of feasible subsets in Table 1 occurs in the middle range—either 3 or 4 items when $c = 6$, exactly 4 items when $c = 7$. However, if a player receives just his or her top-ranked item (1)—the complement of 23456 or 234567 in Table 1—or receives his or her 2^{nd} and 3^{rd} ranked items (23), these small subsets are also feasible. Thus, it seems likely that the players will have several envy-free splits to choose from when there are more than a few items in the contested pile.

Brams and Fishburn [4] derived the following formula for $f(c)$, the number of possible envy-free splits $(S, -S)$ of a contested pile with c items:

$$f(c) = \begin{cases} 2^{c-1} - \dbinom{c}{(c-1)/2} & \text{if } c \text{ is odd} \\ 2^{c-1} - \dbinom{c}{c/2} & \text{if } c \text{ is even} \end{cases}$$

As illustrated in Table 2, $f(c)$ increases exponentially in c:

$c = 1$	$c = 2$	$c = 3$	$c = 4$	$c = 5$	$c = 6$	$c = 7$	$c = 8$	$c = 9$
0	0	1	2	6	12	29	58	130

Table 2 Possible envy-free splits for $c = 1$ to $c = 9$

Notice that, for odd c beginning at $c = 3$, $f(c + 1) = 2f(c)$. Manifestly, the larger c is, the more likely there will be an envy-free split because of the rapidly increasing number of possibilities.

We note that we have accounted only for strict envy-free splits, i.e., $(S, -S)$ wherein A strictly prefers S to $-S$, and B strictly prefers $-S$ to S. Other envy-free splits are possible, wherein one or both players are indifferent between S and $-S$, but they can arise only when the players' preferences satisfy certain conditions.

4 The Expected Size of the Contested Pile

We now turn to the question of how many of the items to be divided can be expected to be contested. Let the total number of items be n, and assume A's strict ranking of the items goes from best (item 1) to worst (item n). Also assume that the $n!$ possible strict rankings of B are equiprobable.[8]

Let $c(n)$ be the expected size of the contested pile. If $n = 1$, it is clear that this single item must be put in the contested pile, so $c(1) = 1$. If $n = 2$, then B either ranks 1 above 2 or 2 above 1, each with probability $\frac{1}{2}$. There are two possibilities:

- Both players' rankings are different, so no items go into the contested pile.
- Both players' rankings are the same, so both items go into the contested pile.

Because both situations occur with probability $\frac{1}{2}$, $c(2) = \frac{1}{2}(0) + \frac{1}{2}(2) = 1$.

Now assume that $n \geq 3$. The probability that B most prefers item 1 is $\frac{1}{n}$; in this case, 1 is put into the contested pile, and the players repeat the procedure to divide a set of $n - 1$ items. With complementary probability $\frac{n-1}{n}$, B's most preferred item is not 1; in this case, A gets 1, B gets her most-preferred item, and the players repeat the procedure on the remaining $n-2$ items. It follows that

$$c(n) = \frac{1}{n}\left[1 + c(n-1)\right] + \frac{n-1}{n}c(n-2). \tag{1}$$

Lemma 1 *Suppose that $c(n-1) = c(n-2) = x$. Then $c(n) = c(n+1) = x + \frac{1}{n}$.*

Proof The lemma follows from equation 1, because

$$c(n) = \frac{1}{n}(1 + x) + \frac{n-1}{n}x = x\left(\frac{1}{n} + \frac{n-1}{n}\right) + \frac{1}{n} = x + \frac{1}{n}.$$

Applying (1) with $n + 1$ replacing n produces

$$c(n+1) = \frac{1}{n+1}\left[1 + x + \frac{1}{n}\right] + \frac{n}{n+1}x = \frac{n + nx + 1 + n^2x}{n(n+1)} = \frac{(n+1) + n(n+1)x}{n(n+1)} = x + \frac{1}{n}$$

and the lemma follows. □

Theorem 3 *If $k \geq 1$, then*

$$c(2k+1) = c(2k+2) = 1 + \frac{1}{3} + \frac{1}{5} + ... + \frac{1}{2k+1}. \tag{2}$$

Proof As noted above, $c(1) = c(2) = 1$. If $k = 1$, then using Lemma 1 with $n = 3$ implies that $c(3) = c(4) = 1 + \frac{1}{3}$. The proof is completed by induction. If equation 2 is true for k, then applying Lemma 1 with $n = 2k + 3$ shows that

$$c(2k+3) = c(2k+4) = c(2k+1) + \frac{1}{2k+3}.$$

□

k=0	c(1)	c(2)	1
k=1	c(3)	c(4)	$1+\frac{1}{3}=1.\dot{3}$
k=2	c(5)	c(6)	$1+\frac{1}{3}+\frac{1}{5}=1.5\dot{3}$
k=3	c(7)	c(8)	$1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}=1.67619...$
k=4	c(9)	c(10)	$1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}=1.78730...$

Table 3 Expected size of the contested pile for $n = 1$ to $n = 10$

Table 3 gives some values of $c(n)$ and illustrates Lemma 1—that $c(n) = c(n + 1)$ when n is odd:

Note that the right side of equation (2) is a variant of the well-known harmonic series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+...$$

which is known to be divergent (i.e., the sum does not approach any finite limit as the number of terms increases without bound).[9] The series on the right side of (2) must be unbounded, for if it had a finite sum, say K, then a term-by-term comparison would show that

$$0<\left[\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+...\right]<\left[1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+...\right]=K.$$

Moreover, a positive series with a finite sum must be absolutely convergent, so the terms of the first two summations below can be rearranged, producing

$$0<\left[1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+...\right]+\left[\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+...\right]=\left[1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+...\right]<2K<\infty,$$

which contradicts the known divergence of the harmonic series. Therefore, the right side of (2) does not approach any limit as k (or $n = 2k + 1$) increases without bound.

In conclusion, the expected size of the contested pile increases without limit as the number of items to be divided increases, but the increase is very slow. Thus, for 9 or 10 items, the expected size of the contested pile is less than 2 (see the last row of Table 3).

To be sure, the calculation in this section depends heavily on the equiprobability assumption, which in practice will almost surely be violated because of a positive correlation between the rankings of A and B. With 9 or 10 items, the number of items in the contested pile might be 4 or 5 instead of about 2.[10] But this is not an unduly large number for which to name minimal bundles, so we think UP will be applicable in such situations.

5 Conclusion

In this paper we have shown how to divide a set of indivisible items between two players in such a way that envy-freeness can, in general, be achieved. Nontrivial envy-free splits will be realized whenever the players have different sets of minimal bundles. In this case, UP implements such splits. As only sets with a certain structure—which we called feasible sets—could ever be part of an envy-free split, we gave a necessary and sufficient condition for any set to be feasible. In addition, we discussed the possible size of the contested pile and various properties of UP.

From a practical standpoint, UP is relatively easy to apply, because it requires that the players indicate only ordinal rather than cardinal preference information—which is required, for example, of

[8] This assumption is equivalent to the impartial culture assumption in models of voting behavior.

[9] The sum of the first n terms is of the same order as $\ln n = \log_e n$, the natural logarithm of n.

[10] In the Panama Canal Treaty dispute between Panama and the United States in the 1970s, a ranking of the 10 issues in the dispute shows that 4 would have been contested under UP. We emphasize, however, that most of the issues were divisible, and compromises were, in fact, reached on several (Raiffa, 1982, ch. 12; Brams and Taylor, 1996, ch. 5).

"adjusted winner" (Brams and Taylor, [6] and [7])—to obtain envy-free splits. While "strict alternation" (Brams and Taylor [7], ch. 2), in which the players take turns choosing items, also uses only ordinal information, the players' positions are not symmetric, making it likely that one will envy the other, especially if their preferences are similar.

We believe that UP would be most valuable in the division of physical items, such as the marital property in a divorce (e.g., a house, car, or boat), which are hard if not impossible to divide or to share. Other examples of all-or-nothing division include two teams choosing players, two political parties choosing cabinet ministries in a coalition government, and the like.

But UP also could be useful in negotiations in which the "items" are issues on which one side or the other might get its way. For example, resolving a border dispute between two countries might require determining which country will have sovereignty over a region, whether there will be a peace-keeping force, and so on, which tend to be all-or-nothing choices and hence not divisible.

Compared to disputes over divisible items like money or land, or disputes in which monetary compensation is possible (Brams and Kilgour [5], Brams [2], ch. 14), disputes involving indivisible issues are more intractable. We believe that the undercut procedure, which is relatively easy to apply, could facilitate agreement in many disputes in which at least some of the items or issues are indivisible and must be awarded, in their entirety, to one or the other of the disputants.

6 Appendix

We showed in section 2 (Corollary 1) that a player's maximin strategy is to rank items sincerely, which guarantees him or her at least 50 percent of the contested pile as long as the players do not have exactly the same minimal bundles. Because a departure from sincerity can lead to a player's getting less than 50 percent of the contested pile, sincerity is a "safe" strategy, especially when information is incomplete.

But if players have complete information about each other's preferences, then a departure from sincerity may be rational. We show this with a simple example in which sincerity is not a Nash equilibrium. Our conclusions are confirmed by the simulation study of Vetschera and Kilgour [11], who assumed preferences are common knowledge.

Assume there are three items to be divided, so there are $3! = 6$ possible rankings, or strategies, that each player can report. These are shown in the 6×6 outcome matrix in Table 4, in which the ordered triple (A, C, B) indicates those items that go to A, those that go into the contested pile (C), and those that go to B, respectively. The superscript n denotes a Nash equilibrium outcome for the associated sincere rankings; underscored outcomes are non-Nash outcomes for sincere rankings. Superscripts A or B denote the player who has a best response to a non-Nash outcome, and a "$*$" a Nash outcome to which there is a best response from a non-Nash outcome (underscored) by A or B.

$B \Rightarrow$ $A \Downarrow$	**123**	**132**	**213**	**231**	**312**	**321**
123	$(-,123,-)^n$	$(2,1,3)^{n^*}$	$(1,3,2)^n$	$\underline{(1,3,2)}^A$	$\underline{(1,2,3)}^B$	$(1,2,3)^n$
132	$(3,1,2)^{n^*}$	$(-,123,-)^n$	$\underline{(1,3,2)}^B$	$(1,3,2)^n$	$(1,2,3)^n$	$\underline{(1,2,3)}^A$
213	$(2,3,1)^n$	$\underline{(2,3,1)}^A$	$(-,123,-)^n$	$(1,2,3)^{n^*}$	$(2,1,3)^n$	$\underline{(2,1,3)}^B$
231	$\underline{(2,3,1)}^B$	$(2,3,1)^n$	$(3,2,1)^{n^*}$	$(-,123,-)^n$	$\underline{(2,1,3)}^A$	$(2,1,3)^n$
312	$\underline{(3,2,1)}^A$	$(3,2,1)^n$	$(3,1,2)^n$	$\underline{(3,1,2)}^B$	$(-,123,-)^n$	$(1,3,2)^{n^*}$
321	$(3,2,1)^n$	$\underline{(3,2,1)}^B$	$\underline{(3,1,2)}^A$	$(3,1,2)^n$	$(2,3,1)^{n^*}$	$(-,123,-)^n$

Table 4 Outcome Matrix for Six Different Rankings of Three Items by A and B

The outcomes shown in Table 4 define a *game form*, because they are products of the players' strategies independent of their preferences. The game form becomes a game when we assume the players have specific preferences, from which we can determine Nash equilibria.

Strategy pairs associated with 24 of the 36 outcomes in the resulting payoff matrix are Nash equilibria if the players choose strategies that coincide with their sincere rankings of the items. As an example, consider the upper left entry of the outcome matrix, in which A and B choose exactly the same sincere strategy, $1P_i2P_i3$ (written as **123** in Table 4). Then all three items go into the contested pile, so the outcome is $(-, 123, -)$.

Assume that each player has a 50-50 chance of getting each item in the contested pile, so each player expects to receive one-half of the total utility in the contested pile. This is better than some allocations and worse that others.[11] In the example, it is not hard to show that neither player can do better by departing from his or her sincere strategy, so these strategies constitute a Nash equilibrium.

Now assume that A continues to choose sincere strategy $1P_A2P_A3$, but B's preferences change, so that B's sincere strategy is $3P_B1P_B2$, giving outcome $(1, 2, 3)$ in row 1, column 5 (underscored in Table 4 and superscripted B). By switching to strategy **132** (column 2), B can effect outcome $(2, 1, 3)$, which is starred and which she prefers because a 50-50 chance of getting item 1 is better than a 50-50 chance of getting item 2 (B gets her best item, 3, in either case; by comparison, A gets his next-best item and a 50-50 chance of getting his best item). Because B can do better by switching from her sincere strategy of **312**, the players' sincere strategies are not in equilibrium.

Given the switch by B, can A, in turn, do better by switching to a different strategy? The answer is "no," because A's sincere strategy associated with outcome $(2, 1, 3)$ at row 1, column 2, is part of a Nash equilibrium.

This is also true of the best responses by B to the non-Nash outcomes superscripted B in each of the other five rows. To these best responses, which are starred, A has no counterresponse. The resulting Nash outcomes are better for B and worse for A than the non-Nash outcomes. Similarly, these same starred outcomes show the best responses by A to the non-Nash outcomes superscripted A.

To summarize, either A or B can benefit by deviating from his or her sincere strategy associated with the 12 underscored sincere outcomes in Figure 4. Improving on a sincere outcome in this way shows the vulnerability of UP to manipulation.

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[11] If A's preference is $1P_A2P_A3$, his ordering from best to worst of the seven possible outcomes satisfies $(1, 2, 3) \succ \{(1, 3, 2), (2, 1, 3)\} \succ (-, 123, -) \succ \{(2, 3, 1), (3, 1, 2)\} \succ (3, 2, 1)$. A strict specification of A's ranking of the outcomes requires further information on A's preferences. Players with different preferences will have analogous rankings of the outcomes; the pure Nash equilibria of the game in Table 4 can be determined from such preferences.