MAKING THE RULES OF SPORTS FAIRER*

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Abstract. The rules of many sports are not *fair*—they do not ensure that equally skilled competitors have the same probability of winning. As an example, the penalty shootout in soccer, wherein a coin toss determines which team kicks first on all five penalty kicks, gives a substantial advantage to the first-kicking team, both in theory and practice. We show that a so-called Catch-Up Rule for determining the order of kicking would not only make the shootout fairer but also is essentially strategyproof. By contrast, the so-called Standard Rule now used for the tiebreaker in tennis is fair. We briefly consider several other sports, all of which involve scoring a sufficient number of points to win, and show how they could benefit from certain rule changes, which would be straightforward to implement.

Key words. Sports rules, fairness, strategyproofness, Markov process, soccer, tennis

AMS subject classifications. 60J20, 91A80, 91A05

1. Introduction.

In this paper, we show that the rules for competition in some sports are not fair. By "fair," we mean that they give equally skilled competitors the same chance to win—figuratively, they level the playing field. Later we will be more precise in defining "fairness."

We first consider knockout (elimination) tournaments in soccer (i.e., football, except in North America), wherein one team must win. We show that when a tied game goes to a penalty shootout, the rules are not fair. On the other hand, the tiebreak in tennis tournaments, when a set is tied at six games apiece, is fair. We briefly comment on the fairness of the rules in other sports, including three racquet sports and volleyball.

But more than pointing a finger at sports whose rules favor one competitor, we analyze in detail two rules—one old (Win-by-Two Rule) and one new (Catch-Up Rule)—and consider other rules as well that can ameliorate unfairness in some sports. As we will show, the lack of fairness arises not because the present rules are *inherently* unfair, always favoring one player, but rather because they involve an element of chance, such as

- which team wins the coin toss in a penalty shootout in soccer and almost invariably elects to kick first;
- which team initially serves in volleyball, in which the team to score 25 points first and be ahead by a margin of at least two points, wins.

We use ideas from fair division and game theory. In game theory, a game is defined by "the totality of the rules that describe it" [18]. (Wittman [19] offers an intriguing discussion of "efficient rules," which are often used as substitutes for economic markets in sports and other activities.) In almost all competitive sports, the rules allow for some element of chance, such as who gets to move first. In the final round of a golf tournament, it is in fact the players who get to play last—the order is not fixed by the rules—who know what they must score to win. This knowledge may help them decide whether to try a risky shot or not, which is information the first players to finish do not have.

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In the National Football League, if a game ends in a tie, a coin toss determines which team decides whether to kick or receive in the overtime period. Almost always, the winner of the coin toss elects to receive, which statistics indicate gives it a substantial advantage of winning the game. Che and Hendershott [7, 8] proposed that the teams bid on the yard line that would make them indifferent between kicking and receiving; Brams and Sanderson [6] and Granot and Gerschak [11] further analyzed this rule and discussed the extension of bidding to soccer and chess to render competition fairer. In this paper, we propose very different solutions to the fairness problem in sports when bidding may be deemed infeasible or unacceptable.

Handicaps are sometimes used in these situations to make contests more competitive. In golf, for example, if A has a handicap of four strokes and B has a handicap of two strokes, then A, with a score of 80, can beat B, with a score of 79 (lower scores win in golf), because

$$80 - 4 = 76 \quad < \quad 79 - 2 = 77.$$

That is, when the handicaps are subtracted to give net scores, B beats A 76 to 77. Thereby the handicaps turn B from a loser into a winner.

Handicaps in sports take different forms. In horse racing, horses may carry additional weight according to their speed in past performances, with the fastest horse carrying the most weight (other factors also matter, such as post position and the jockey). Handicapping is also done by starting horses at different points, with the fastest horse having the greatest distance to run in order to win.

In general, handicapping gives an advantage to weaker competitors—as compensation for their lower level of skill—to equalize the chances that all competitors can win. Handicapping is used in a variety of sports and games, including bowling, chess, Go, sailboat racing, baseball, basketball, football (American), and track and field events, where it serves as the basis for wagering on the outcomes of these contests. Thus, a weak player or team can beat a strong one if the point spread that the strong one must win by is sufficiently large.

In the subsequent analysis, we consider competitions in which handicapping may not be feasible or desirable. Instead, handicapping, if any, occurs *in the course of play*. More specifically, the Catch-Up Rule takes into account the results of competition in the preceding contest: Players or teams that do worse in the preceding contest are afforded the opportunity to catch up. (This idea is incorporated in a game, Catch-Up, which is analyzed in Isaksen, Ismail, Brams, and Nealen [12]. For a demo version of this game, see http://game.engineering.nyu.edu/projects/catch-up/.) Greater fairness can also be engendered by the Win-by-Two Rule, which precludes a player or team from winning by just one point and can minimize, or even eliminate, the role of chance.

The paper is organized as follows. In Section 2, we define the Catch-Up Rule and a related rule, the Behind First, Alternating Order Rule, and apply them to penalty kicks in soccer, the world's most widely played and popular sport. In Section 3, we show how these rules tend to equalize the probability of each side winning, compared with what we call the Standard Rule, based on a coin toss. The Standard Rule, which varies from sport to sport, determines in soccer which team kicks first on every round of the penalty shootout. This rule gives the team that wins the coin toss, and generally chooses to kick first, a decided edge.

In Section 4, we consider the situation when, after five penalty kicks, the teams remain tied. Then the outcome is determined by a form of *sudden death*, whereby the first team to score a goal on a round without the other team scoring, wins.

In this infinite-horizon situation (there is no definite termination), we analyze the probability of each side winning under the Standard Rule, the Catch-Up Rule, and the Behind First, Alternating Order Rule in situations when teams are equally skilled and when they are not. We also consider the incentive that a team might have to try to manipulate the outcome by not making a maximal effort to win a point, either when it kicks or when its opponent kicks, and show that, for all practical purposes, the Catch-Up Rule is incentive compatible or strategyproof.

In Section 5, we turn to tennis, one of the most popular two-person sports, showing that the Standard Rule in the tiebreak, when a set is tied at six games apiece, is fair, primarily because of the alternation in serving and the Win-by-Two rule. In Section 6, we briefly consider other sports and games and comment on the fairness of their rules. In Section 7, we offer some concluding thoughts on the practicality of changing the rules of sports to render them fairer.

2. The Catch-Up Rule, the Behind First, Alternating Order Rule, and Their Application to Soccer. Suppose that a soccer game, after regulation play and sometimes extra-time periods, ends in a tie. Because soccer is a low-scoring game, ties, which may be as low as 0-0, are common. To break ties in a knockout tournament, there is a penalty-kick shootout, whereby a different player from each team, over five rounds, is given the chance to score a goal from 11 meters in front of the goal line, which is defended by the other team's goal-keeper.

The team that scores more goals in the shootout wins. If the scores are tied at the end of the five rounds of the shootout, then the game goes to sudden death. If one team (say, A) gains an insurmountable lead over the other (B), the shootout terminates early: Even if B scores on all the remaining rounds and A does not, A would still win. In particular, if A leads 3-0 or 4-2 in the shootout, there is no possibility that B can win or tie, even if it scores and A does not score on the remaining rounds.

The reason for having five rounds in penalty shootouts is to ensure, insofar as seems reasonable, that the stronger team wins with a high probability. If play were immediately to go to sudden death, luck would play an unduly large role, making it more chance than skill that one team happens to score, and the other does not, on a single round. But over five rounds there is less variance in the probability that the better team will win. But "better" in this case means a team's ability, in several two-person competitions between a kicker and a defender, to score and to prevent the other team from scoring. This ability, however, may have little relationship to the ability of an 11-person team to win in regular play, which is why penalty shootouts are unpopular with many fans.

Which team kicks first on each of the five rounds of kicks, when each team has one kick, is determined by a coin toss. This is the Standard Rule in penalty shootouts. The team that wins the toss almost always elects to kick first, because doing so is generally considered advantageous. It puts psychological pressure on the team that kicks second, especially if the team kicking first scores on its kick [15].

In major tournaments between 1970 and 2013, the team that kicked first won the penalty shootout 60.6% of the time [15], giving it a substantial 3:2 advantage, or 50% greater probability, of winning the tied game. Earlier Apesteguia and Palacios-Huerta [2] found a 60.5% first-mover advantage, using a dataset of 269 shootouts from 1970 to 2008; later, Kocher, Lenz and Sutter [14] observed a 53.3% advantage, using a dataset of 540 shootouts from 1970 to 2003. The dataset reported in [15] used 1001 shootouts from 1970 to 2013. When coaches and players were asked in a survey about whether they would choose to go first or second if they won the coin toss, more than

90% said they would go first [2]. Clearly, the advantage of kicking first is not only perceived, but it is, in fact, large.

A partial solution to the first-kicker bias would be to use a coin toss on each round of the penalty shootout to determine the order of kicking. Like the present rule, this would be ex ante but not ex post fair. For example, a team that wins three tosses in a row, which is not a rare event, would get a nice break from this rule, whereas our Catch-Up Rule, as discussed next, would halt a string of successes because of such a break.

2.1. Catch-Up Rule. The Catch-Up Rule is designed to mitigate this bias. To make it applicable to sports other than soccer, we formulate it below for all sports with multiple contests.

More specifically, we assume there is a series of contests (penalty kicks in soccer) in which, in each contest, there is an advantaged and a disadvantaged player or team (the advantaged team in soccer is the team that kicks first in a round). If a player or team *wins* a contest, call it W; if it *loses*, call it L. When no player or team wins or loses a contest, call the contest *unresolved* (U). Play proceeds as follows.

1. In the first contest, a coin is tossed to determine which player or team is advantaged and which is disadvantaged.
2. In this contest and every subsequent contest in which one player or team becomes W and the other L, the player or team that was L becomes advantaged in the next contest.
3. If a contest is U because both teams become L or both become W, the player or team that was advantaged in it becomes the other player or team in the next contest.

In soccer, the contests are rounds, in which each team has one kick. In the first round, one team is advantaged (by kicking first). If it is successful (W) and the other team is not (L), rule 2 says that L becomes advantaged on the next round (whether it was advantaged or disadvantaged in the current round). If both teams on a round are either successful or unsuccessful in scoring a point, neither team becomes L or W—the contest is U. Rule 3 says that the team that was advantaged in a U round becomes disadvantaged in the next round.

Let the two teams in soccer be X and Y. In the first round, assume that X wins the coin toss and, therefore, is advantaged. If the contest turns out to be U, Y will be advantaged on the next round; Y will also be advantaged if X wins on the first round. Only if X loses and Y wins on the first round will X be advantaged in the next round.

In the subsequent analysis, we assume that X wins the coin toss, so it kicks first. To illustrate our analysis in a simple case, assume that there are just two rounds in the penalty shootout, allowing X and Y two kicks each.

In Figure 1, we illustrate all the possible states of a two-round penalty shootout, wherein the order in which we write X and Y indicates which team shoots first, and which second, according to the Catch-Up Rule (e.g., XY indicates that X shoots first and Y second). The numbers in parentheses, (I-J), give the scores of X and Y, respectively, in that state. The first-round states are unshaded and the second-round states are shaded.

The shootout starts in the center, XY(0-0), in which X kicks first, Y second, and the score is 0-0. There are four cases of continuation, whereby in the first round X and Y both score (++), X scores and Y does not (+−), X does not score and Y does (−+), and neither players scores (−−). Arrows point to these four states from

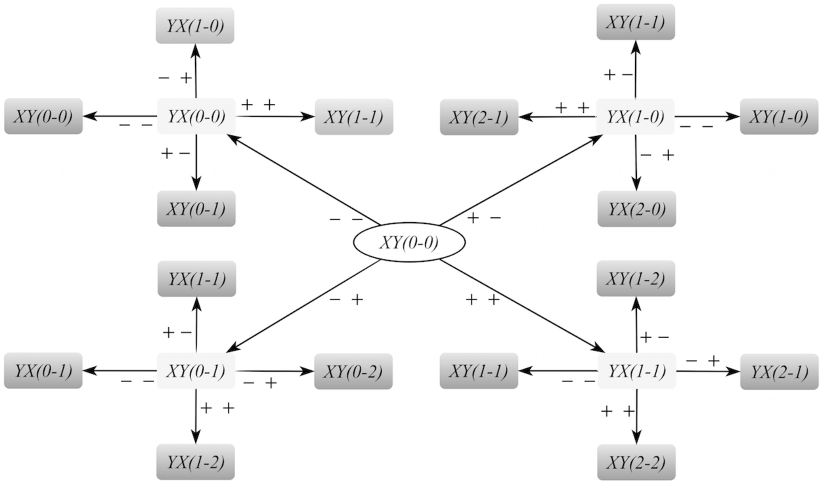

FIG. 1. *Possible States of Catch-Up over Two Rounds in the Penalty Shootout*

XY(0-0).

Emanating from each of these four states are four more arrows, pointing to four shaded states, which constitute the second round, in which both, one, or neither team scores. Observe that if both teams score on both rounds, play ends at XY(2-2) in the lower right shaded state, in which X kicks first because Y did at YX(1-1). This order also holds at XY(0-0) in the upper left, in which neither player scores on both rounds.

Before reaching either of these two shaded states, the order of kicking switches to YX—at YX(1,1) in the first case, and YX(0,0) in the second case. Only when X fails to score and Y does score at XY(0-1) in the first round does the order of kicking not switch (it stays XY), because by rule 2 X is still advantaged since it did not score when Y did.

2.2. Behind First, Alternating Order Rule. The Catch-Up Rule differs from the "behind first, alternating order" mechanism of Anbarci et al. [1], which depends on the current score: If one team is behind, it kicks first on the next round; if the score is tied, the order of kicking alternates (i.e., switches from the previous round). By comparison, the Catch-Up Rule depends only on the performances of the teams on the previous round.

To illustrate the difference between the two mechanisms, suppose that X is ahead by one point. Under the Behind First, Alternating Order Rule, Y will be advantaged next since it is behind. But under the Catch-Up Rule, X could be advantaged next. (For example, if Y was previously behind by two points and reduced the deficit to one point by scoring when X did not, then X would be advantaged next, even though Y is still behind.) Thereby, the Catch-Up Rule allows for the possibility that a player who is ahead will be advantaged; it does not automatically confer an advantage on the player who is behind, as does the Behind First, Alternating Order Rule.

There are other differences between the two mechanisms. Anbarci et al. [1] model the optimal spot for the kicker to aim at to balance the probability of the goalie stopping a goal (if the ball is aimed close to the center of the net) and the probability of the ball going outside the net (if the ball is aimed close to a post on the side). By contrast, we offer no such model but assume instead that the probability of a successful kick depends only on whether the kicker kicks first or second on a round, implicitly assuming that where to aim has been optimally chosen and not modeling this aspect of the game. We believe that this is a reasonable assumption in the penalty shootout, especially because each kicker gets only one try. We spell out later the Markovian decision process that the Catch-Up Rule assumes, but first we define some terms we will employ in our probabilistic analysis.

3. Probabilistic Analysis of the Penalty Shootout. Let $p_i^Z \in [0,1]$ be the probability that team Z (X or Y) is successful if its kicker on round i ($i = 1, 2, ...$) kicks first, and $q_i^Z \in [0,1]$ be the probability that its kicker on round i is successful if it kicks second, where kicks by each team are assumed to be independent events. We omit the superscripts (subscripts) when we suppose that the scoring probability of a player is independent of the team (round).

Let $P^r(X)$ denote the probability of X winning (by at least one point), $P^r(Y)$ the probability of Y winning, and $P^r(T)$ the probability of a tie (T) over r rounds. We calculate separately these probabilities under the Standard Rule and the Catch-Up Rule.

3.1. Standard Rule. Under this rule, because X won the coin toss, it kicks first on every round. To illustrate, when $r = 2$, X can win with scores of 2-0, 2-1, or 1-0, whose three probabilities, respectively, are given by the three bracketed expressions below:

$$P^2(X) = [p_1^X(1-q_1^Y)p_2^X(1-q_2^Y)] + [p_1^X(1-q_1^Y)p_2^X q_2^Y + p_1^X q_1^Y p_2^X(1-q_2^Y)]$$
$$+[p_1^X(1-q_1^Y)(1-p_2^X)(1-q_2^Y) + (1-p_1^X)(1-q_1^Y)p_2^X(1-q_2^Y)].$$

The first summand in brackets gives the probability that X scores and Y does not on both rounds. In the second summand, X scores on both rounds but Y does not on one round (either the first or the second). In the third summand, X scores on one round (either the first or the second) but Y does not on both rounds.

Analogously, by interchanging (p_1^X, p_2^X) and (q_1^Y, q_2^Y) in the formula for $P^2(X)$, Y can win with scores of 0-2, 1-2, and 0-1, given by the three bracketed expressions below:

$$P^2(Y) = [(1-p_1^X)q_1^Y(1-p_2^X)q_2^Y] + [(1-p_1^X)q_1^Y p_2^X q_2^Y + p_1^X q_1^Y(1-p_2^X)q_2^Y]$$
$$+[(1-p_1^X)q_1^Y(1-p_2^X)(1-q_2^Y) + (1-p_1^X)(1-q_1^Y)(1-p_2^X)q_2^Y].$$

Finally, the probability of a tie (T), which may be 0-0, 1-1, or 2-2, is given, respectively, by the three probabilities in the three bracketed expressions below:

$$P^2(T) = [(1-p_1^X)(1-q_1^Y)(1-p_2^X)(1-q_2^Y)] + [p_1^X q_1^Y(1-p_2^X)(1-q_2^Y)$$
$$+p_1^X(1-q_1^Y)(1-p_2^X)q_2^Y + (1-p_1^X)q_1^Y p_2^X(1-q_2^Y) + (1-p_1^X)(1-q_1^Y)p_2^X q_2^Y]$$
$$+[(p_1^X q_1^Y p_2^X q_2^Y].$$

The first and third expressions give the probabilities, respectively, that both players do not score, or do score, on both rounds. The second expression reflects the fact that X and Y can each score, or not score, one point on either the first or the second round, giving $2 \times 2 = 4$ different ways in which a 1-1 tie can be achieved. As a check on these formulas, one can verify that

$$P^2(X) + P^2(Y) + P^2(T) = 1.$$

To illustrate the advantage that the Standard Rule confers on X, assume $p_1^X = p_2^X = 3/4$ and $p_1^Y = p_2^Y = 2/3$, favoring X on each round by the ratio $(3/4)/(2/3)$, which gives X a 9:8 advantage, or a $1/8 = 12.5\%$ greater probability than Y of winning on each round. Over the two rounds, this advantage is significantly magnified:

$$(1) \qquad P^2(X) = \frac{51}{144} = 0.354; \quad P^2(Y) = \frac{32}{144} = 0.222; \quad P^2(T) = \frac{61}{144} = 0.424.$$

The first two probabilities give X a 51:32 advantage, or a $19/32 = 59.4\%$ greater probability than Y, of winning. Notice, however, that $P^2(T)$ is the biggest of the three probabilities; we will show later how sudden death, when there is a tie, reallocates this probability to $P^2(X)$ and $P^2(Y)$.

The following proposition gives formulas for the three probabilities when the kickers of X and Y are equally skilled, so p and q depend only on whether X or Y kicks first or second on a round—not the player for X or Y who kicks on that round (i.e., p and q are not kicker-dependent).

PROPOSITION 3.1. *Assume there are r rounds, on each of which the advantaged team (first kicker) has probability p of scoring and the disadvantaged team (second kicker) has probability q of scoring. If* X *kicks first,*

$$P^r(X) = \sum_{i=0}^{r-1} \sum_{j=0}^{r-i-1} \binom{r}{r-i} \binom{r}{j} p^{r-j}(1-p)^i q^j (1-q)^{r-j},$$

$$P^r(Y) = \sum_{i=0}^{r-1} \sum_{j=0}^{r-i-1} \binom{r}{r-i} \binom{r}{j} q^{r-j}(1-q)^i p^j (1-p)^{r-j},$$

$$P^r(T) = \sum_{j=0}^{r} \binom{r}{j} \binom{r}{j} p^j (1-p)^{r-j} q^j (1-q)^{r-j}.$$

Proof. The number of ways in which X can win by scoring on m out of the r rounds is the number of ways in which Y's score is less than m, which is $\binom{r}{m}\binom{r}{m-i}$ for all values of $1 < m \leq r$ and $1 \leq i \leq m$. The double summation on the left side of $P^r(X)$ gives all ways in which X can beat Y—1-0, 2-0, 2-1, ..., r-$(r-1)$—multiplied by the corresponding probability for each of the ways. $P^r(Y)$ interchanges p and q in the formula for $P^r(X)$. $P^r(T)$ reflects the fact that X and Y obtain the same score, with the single summation over all ways in which this can occur. ∎

3.2. Catch-Up Rule. We next consider the effects of the Catch-Up Rule over two rounds of penalty kicks. X and Y can win, lose, or tie with the same scores as under the Standard Rule. But the ways these scores are realized are different: The ordering of kicking on each round is endogenous (i.e., dependent on the previous round) rather than exogenous (i.e., fixed in advance). We list below the three ways that X can win with scores of (i) 2-0, (ii) 2-1, and (iii) 1-0:

(i) *2-0:* X *scores on both rounds while* Y *fails to score on both.* On the first round, X succeeds and Y fails with probability $p_1^X(1-q_1^Y)$. On the second round, Y kicks first and fails while X kicks second and succeeds with probability $(1-p_2^Y)q_2^X$. The joint probability of this outcome over both rounds is $p_1^X(1-q_1^Y)(1-p_2^Y)q_2^X$.

(ii) *2-1:* X *scores on both rounds while* Y *fails to score on one of these rounds.* There are two cases:

(a) Assume Y scores on the first round. On this round, both players succeed with probability $p_1^X q_1^Y$. On the second round, Y kicks first and fails, after which X succeeds, with probability $(1-p_2^Y)q_2^X$. The joint probability over both rounds is $p_1^X q_1^Y(1-p_2^Y)q_2^X$.

(b) Assume Y scores on the second round. On the first round, X succeeds and Y fails with probability $p_1^X(1-q_1^Y)$. On the second round, Y kicks first and succeeds, after which X succeeds, with probability $p_2^Y q_2^X$. The joint probability over both rounds is $p_1^X(1-q_1^Y)p_2^Y q_2^X$.

Thus, the probability that the outcome is 2-1 is

$$p_1^X q_1^Y(1-p_2^Y)q_2^X + p_1^X(1-q_1^Y)p_2^Y q_2^X.$$

(iii) *1-0:* X *scores on one round while* Y *fails to score on both rounds.* There are two cases:

(a) Assume X scores on the first round. On this round, X succeeds and Y fails with probability $p_1^X(1-q_1^Y)$. On the second round, Y kicks first and fails, after which X fails, with probability $(1-p_2^Y)(1-q_2^X)$. The joint probability over both rounds is $p_1^X(1-q_1^Y)(1-p_2^Y)(1-q_2^X)$.

(b) Assume X scores on the second round. On the first round, both players fail with probability $(1-p_1^X)(1-q_1^Y)$. On the second round, Y kicks first and fails, after which X succeeds, with probability $(1-p_2^Y)q_2^X$. The joint probability over both rounds is $(1-p_1^X)(1-q_1^Y)(1-p_2^Y)q_2^X$.

Thus, the probability that the outcome is 1-0 is

$$p_1^X(1-q_1^Y)(1-p_2^Y)(1-q_2^X) + (1-p_1^X)(1-q_1^Y)(1-p_2^Y)q_2^X.$$

Summing the probabilities that X wins in cases (i), (ii), and (iii), given by each of the summands in brackets below, yields

$$P^2(X) = [p_1^X(1-q_1^Y)(1-p_2^Y)q_2^X] + [p_1^X q_1^Y(1-p_2^Y)q_2^X + p_1^X(1-q_1^Y)p_2^Y q_2^X]$$
$$+[p_1^X(1-q_1^Y)(1-p_2^Y)(1-q_2^X) + (1-p_1^X)(1-q_1^Y)(1-p_2^Y)q_2^X .]$$

An analogous for formula for $P^2(Y)$ for cases (i), (ii), and (iii), given by each of the summands, yields

$$P^2(Y) = [(1-p_1^X)q_1^Y(1-p_2^X)q_2^Y] + [p_1^X q_1^Y p_2^Y(1-q_2^X) + (1-p_1^X)q_1^Y p_2^X q_2^Y]$$
$$+[(1-p_1^X)q_1^Y(1-p_2^X)(1-q_2^Y) + (1-p_1^X)(1-q_1^Y)p_2^Y(1-q_2^X)],$$

and, as with the Standard Rule, $P^2(T) = 1 - P^2(X) - P^2(Y)$.

For the values of $p = 3/4$ and $q = 2/3$ assumed earlier, we obtain

$$(2) \quad P^2(X) = \frac{41}{144} = 0.284; \quad P^2(Y) = \frac{39}{144} = 0.270; \quad P^2(T) = \frac{64}{144} = 0.444.$$

Comparing the values of (1) and (2), we see that the Catch-Up Rule tends to equalize the values of $P^2(X)$ and $P^2(Y)$ over those given by the Standard Rule. More specifically, X is favored in $2/39 = 5.1\%$ more cases than Y, whereas recall that under the Standard Rule X was favored in 59.4% more cases than Y. Thus, the Catch-Up Rule cuts the bias in favor of X by a factor of more than ten.

How does the number of rounds affect the advantage of X over Y? Consider, for example, the trivial case for one round when $p = 3/4$ and $q = 2/3$, in which the Standard Rule and the Catch-Up Rule give the same probabilities,

$$P^1(X) = p(1-q) = 0.250; \quad P^1(Y) = (1-p)q = 0.167; \quad P^1(T) = 0.583,$$

because, without a second round, the Catch-Up Rule cannot change the order of kicking from that of the Standard Rule.

As the number of rounds increases from one to five, we have calculated $P^r(X)$, $P^r(Y)$, and $P^r(T)$ for the Standard Rule, the Behind First, Alternating Order Rule, and the Catch-Up Rule. (Recall from Section 2 that the Behind First, Alternating Order Rule is that if a team is behind in scoring, it will be advantaged—whether or not it was on the previous round—but if there is a tie, the team previously advantaged switches to the other team. This rule may be thought of as score-dependent, whereas the Catch-Up Rule is performance-dependent: The advantaged team is the one which lost on the previous round—independent of the score, so the history of play prior to the previous round is irrelevant—but if both teams score or do not on a round, the advantaged team switches from the previous round.) We have written a computer program to iterate the process to five rounds for the calculation given below.

It is worth pointing out that our probabilistic calculations are not affected if a shootout ends early—before five rounds are completed—should one team attain an insurmountable lead (see Section 2). This is so because a victory that occurs before the completion of five rounds would be a victory after the completion of five rounds and, consequently, would have the same probability of occurrence.

Apesteguia and Palacios-Huerta [2, p. 2558] present empirical probabilities, based on a dataset of 269 penalty shootouts, that X and Y lead at the end of r rounds, with r ranging from round 1 to round 5,

$$[P^r(X), P^r(Y)]_{i=1}^5 = [(0.20, 0.13), (0.27, 0.20), (0.35, 0.23), (0.40, 0.27), (0.46, 0.27)],$$

in which each pair in the five-element sequence gives the probability of the first-kicking team (X) and the second-kicking (Y) team, respectively, winning on that round. The probability of a tie at the end of each round is the complement of these probabilities (e.g., at the end of round 5, the probability of a tie is $1 - (0.46 + 0.27) = 0.27$).

Our starting point is the empirical probabilities that X and Y, respectively, score a point on each of rounds 1 through 5, as given by the following sequence pairs [2, p. 2558]:

$$[p_i, q_i]_{i=1}^5 = [(0.79, 0.72), (0.82, 0.77), (0.77, 0.64), (0.74, 0.68), (0.74, 0.67)].$$

These probabilities differ from the empirical probabilities in the preceding paragraph, which were cumulative—they gave the probabilities that X and Y lead after each round instead of the probabilities that they score on each round.

Let X's and Y's probabilities on each round be the empirical probabilities given by $[p_i, q_i]_{i=1}^5$. The theoretical results for shootouts lasting from one to five rounds are shown in Table 1.

As the number of rounds r increases from 1 to 5, we summarize below how the values of $P^r(T)$, $P^r(X)$, and $P^r(Y)$ change for the Standard Rule, the Behind First, Alternating Order Rule, and the Catch-Up Rule:

1. $P^5(X)$ and $P^5(Y)$ almost duplicate the empirical frequency with which X and Y lead at the end of round 5 under the Standard Rule [2, p. 2558].
2. For all rules, $P(T)$ *decreases* by a factor of about 1.9 or more—from 0.628 to (i) 0.274 for the Standard Rule, (ii) 0.323 for the Behind First, Alternating Order Rule, and (iii) 0.289 for the Catch-Up Rule.
3. For the Standard Rule, $P(X)$ *increases* by a factor of 2.1 (0.466/0.221), whereas $P(Y)$ *increases* by a factor of 1.7 (0.260/0.151).
4. For the Behind First, Alternating Order Rule $P(X)$ *increases* by a factor of 1.6 (0.364/0.221), whereas $P(Y)$ *increases* by a factor of 2.1 (0.313/0.151).

		1 Round	2 Rounds	3 Rounds	4 Rounds	5 Rounds
	$P(X)$	0.221	0.309	0.403	0.483	0.466
Standard Rule	$P(Y)$	0.151	0.210	0.228	0.250	0.260
	$P(T)$	0.628	0.481	0.369	0.312	0.274
Behind First	$P(X)$	0.221	0.266	0.334	0.346	0.364
Alternating	$P(Y)$	0.151	0.241	0.256	0.299	0.313
Order Rule	$P(T)$	0.151	0.492	0.410	0.355	0.323
	$P(X)$	0.221	0.266	0.353	0.362	0.385
Catch-Up Rule	$P(Y)$	0.151	0.241	0.270	0.310	0.326
	$P(T)$	0.628	0.492	0.377	0.328	0.289

TABLE 1
Probability That X *Wins,* Y *Wins, or There Is a Tie after 1-5 Rounds for the Three Rules, Based on the Empirical Probabilities Given by* $[p_i, q_i]_{i=1}^5$

5. For the Catch-Up Rule, $P(X)$ *increases* by a factor of 1.7 (0.385/0.221), whereas $P(Y)$ *increases* by a factor of 2.2 (0.326/0.151).

Clearly, as $P^r(T)$ decreases, $P^r(X)$ and $P^r(Y)$ tend to diverge under the Standard Rule but converge under the Behind First, Alternating Order Rule and the Catch-Up Rule. More specifically, the latter two rules give Y the ability almost to catch up to X by giving it more opportunity to kick first, whereas the Standard Rule, by never allowing this, increases the disparity between the advantaged and the disadvantaged teams.

In relative terms, the ratio, $P(X)/P(Y)$, is 1.5 on the first round for all three rules. While it increases to 1.8 for the Standard Rule after five rounds, it hovers around 1.2 for the Behind First, Alternating Order Rule and the Catch-Up Rule from rounds one to five. Hence, if there is more than one round, these two rules quickly reduce the big advantage that X enjoys over Y, whereas the Standard Rule increases X's advantage.

We next calculate $P^r(X)$, $P^r(Y)$, and $P^r(T)$ in a penalty shootout in which the teams are differently skilled. We again start from the empirical probabilities that X and Y, respectively, score a point on each of rounds 1 through 5 given by $[p_i, q_i]_{i=1}^5$.

Suppose that there is a good team (G) and a bad team (B). Let G's probabilities on each round be the aforementioned empirical probabilities, and B's probabilities be 5 and 10 percentage points lower, respectively, than G's:

5% Lower : $[(0.74, 0.67), (0.77, 0.72), (0.72, 0.59), (0.69, 0.63), (0.69, 0.62)]$

10% Lower : $[(0.69, 0.62), (0.72, 0.67), (0.67, 0.54), (0.64, 0.58), (0.64, 0.57)]$.

The results for penalty shootouts after five rounds, where G is either X (i.e., shoots first) or Y (shoots second), are shown in Table 2.

We summarize below how the values of $P^5(T)$, $P^5(X)$, and $P^5(Y)$ change for the Standard Rule, the Behind First, Alternating Order Rule, and the Catch-Up Rule:

- **Standard Rule:** If G starts, $P^5(X)$ is greater than $P^5(Y)$ by a factor of 2.6 (0.540/0.210) in Case 1 (5%) and by a factor of 3.7 (0.609/0.166) in Case 2 (10%). If G starts second, $P^5(Y)$ is about 7 percentage points lower (0.328 vs. 0.398) than $P^5(X)$ in Case 1—despite being the better team—and about 6 percentage points higher (0.398 vs. 0.335) in Case 2.

		G Is X against B (5% Lower)	**G Is Y against B (5% Lower)**	**G Is X against B (10% Lower)**	**G Is Y against B (10% Lower)**
	$P^5(X)$	0.540	0.398	0.609	0.335
Standard Rule	$P^5(Y)$	0.210	0.328	0.166	0.398
	$P^5(T)$	0.251	0.274	0.224	0.267
Behind First	$P^5(X)$	0.435	0.303	0.505	0.248
Alternating	$P^5(Y)$	0.258	0.384	0.209	0.456
Order Rule	$P^5(T)$	0.307	0.313	0.285	0.296
	$P^5(X)$	0.458	0.322	0.528	0.265
Catch-Up Rule	$P^5(Y)$	0.269	0.399	0.219	0.471
	$P^5(T)$	0.273	0.280	0.253	0.264

Table 2

Probability That X *Wins,* Y *Wins, or There Is a Tie after 5 Rounds When Teams Are Differently Skilled*

- **Behind First, Alternating Order Rule:** If G starts first, $P^5(X)$ is greater than $P^5(Y)$ by a factor of 1.7 (0.435/0.258) in Case 1 (5%) and by a factor of 2.4 (0.505/0.209) in Case 2 (10%). If G starts second, $P^5(Y)$ is greater than $P^5(X)$ by a factor of 1.3 in Case 1 and 1.8 in Case 2.
- **Catch-Up Rule:** If G starts first, $P^5(X)$ is greater than $P^5(Y)$ by a factor of 1.7 (0.435/0.258) in Case 1 (5%) and by a factor of 1.9 (0.505/0.259) in Case 2 (10%). If G starts second, $P^5(Y)$ is greater than $P^5(X)$ by a factor of 1.2 (0.399/0.322) in Case 1 and by a factor of 1.8 (0.471/0.265) in Case 2.

Clearly, both the Behind First, Alternating Order Rule and the Catch-Up Rule give a substantial boost to G, whether it is X or Y. The Standard Rule gives G even more of a boost if it is X, but if it is Y, G's skill may not be sufficiently high to overcome the disadvantage of being Y. These results are consistent with those given in Table 1, which showed that as the number of rounds increases for equally skilled teams, the Standard Rule widens the gap between being X or Y, whereas the Behind First, Alternating Order Rule and the Catch-Up Rule narrow it.

After five rounds, as shown in Table 2, there is a substantial probability (25% to 47%) that a penalty shootout will end in a tie. As we showed in Table 1, the range in the probability of a tie is narrower for equally skilled teams (27% to 39%) but still substantial. Because one must invoke sudden death to determine a winner in these cases, we turn next to their analysis and also consider the possibility of manipulation of the rules.

4. Sudden Death and Strategic Manipulation in the Penalty Shootout. After five rounds of a shootout, the six states in which X and Y can be tied are 0-0, 1-1, 2-2, 3-3, 4-4, or 5-5. When sudden death is applied to break a tie, the first team to score a point on a round when the other team does not become the winner.

If there is a tie after five rounds, under the Standard Rule, X will kick first on every round until the tie is broken. X will win on round i with probability $p_i^X(1-q_i^Y)$; if it fails, it can still win subsequently if both players either score or do not score, which occurs with probability $[p_i^X q_i^Y + (1-p_i^X)(1-q_i^Y)]$.

For simplicity of exposition, we omit the subscripts and superscripts from the subsequent probabilities. For both the Standard Rule and the Catch-Up Rule, let $W(X)$ be the probability that X wins in sudden death; the probability that Y wins is the complement, $W(Y) = 1 - W(X)$.

For the Standard Rule (S), X will win with probability $W^S(X)$, which gives the

		1 Round	2 Rounds	3 Rounds	4 Rounds	5 Rounds
Standard Rule	Q(X)	0.600	0.608	0.618	0.628	0.637
Catch-Up Rule	Q(X)	0.526	0.516	0.518	0.513	0.514

TABLE 3
Probability That X *Wins, with Sudden Death, after 1 to 5 Rounds when* $p = 3/4$ *and* $q = 2/3$ *for the Standard Rule and the Catch-Up Rule*

following recursion:

$$W^S(X) = p(1-q) + [pq + (1-p)(1-q)]W^S(X).$$

Solving for $W^S(X)$ yields

$$W^S(X) = \frac{p(1-q)}{p+q-2pq}.$$

For the Catch-Up Rule, X will win on the first round with probability $p(1-q)$; if that fails, it can win subsequently if both players either score or do not score, which occurs with probability $[pq + (1-p)(1-q)]$. Because the score remains tied, and X kicked first, by rule (3) of the Catch-Up Rule, Y will kick first next (see Section 2). Y, now in the same position as X was at the outset, will win with the same probability, $W^C(X)$. Consequently, X will win with probability $[1 - W^C(X)]$—that is, the probability Y will not win—which gives the following recursion:

$$W^C(X) = p(1-q) + [pq + (1-p)(1-q)][1 - W^C(X)].$$

Solving for $W^C(X)$ yields

$$W^C(X) = \frac{1-q+pq}{2-p-q+2pq}.$$

For $p = 3/4$ and $q = 2/3$, $W^S(X) = 3/5 = 0.600$. Under the Catch-Up Rule, by contrast, $W^C(X) = 10/19 = 0.526$, which is substantially closer to 50%, rendering X and Y more competitive. We forgo calculations for the Behind First, Alternating Order Rule, because this rule coincides with the Catch-Up Rule in sudden death.

But these results hold only if the penalty shootout goes to sudden death. More likely, the shootout will be resolved by round five, or possibly on an earlier round if one team jumps ahead of the other in successful kicks. To foreclose the possibility of a tie in a knockout tournament, we use the sudden death formulas, $W^S(X)$ and $W^C(X)$, to apportion the tied probability, $P^r(T)$, to $P^r(X)$ and $P^r(Y)$, on each round r.

For example, as we showed in Section 3, after two rounds, the probability of a tie under the Standard Rule is $P^2(T) = 61/144 = 0.424$. Because $W^S(X) = 3/5 = 0.600$ and $W^S(Y) = 2/5 = 0.400$, X and Y's probabilities of winning with sudden death will be augmented by their apportioned tied probabilities—$(61/144)(3/5) = 183/720 = 0.254$ for X and $(61/144)(2/5) = 122/720 = 0.169$ for Y—giving the following revised (rounded) probabilities of winning over two rounds that incorporate sudden death:

$$Q^2(X) = 0.354 + 0.254 = 0.608; \quad Q^2(Y) = 0.222 + 0.169 = 0.392.$$

These and the other probabilities for penalty shootouts lasting five or fewer rounds, where $p = 3/4$ and $q = 2/3$, are shown in Table 3 for $Q^r(X)$. Under the

Standard Rule, as r increases from 1 to 5, $Q^r(X)$ increases from 60% to about 64%, which is close to the empirical bias favoring X. By contrast, under the Catch-Up Rule, $Q^r(X)$ stays close to 0.500, so it tends to equalize the probabilities that X and Y win.

There is a small odd-even effect for both rules. Most salient is that after five rounds, and with sudden death if the shootout is still unresolved, the Catch-Up Rule reduces the Standard Rule's bias favoring X from 64% to 51%, making the contest essentially even-steven, though X kicks first on the first round.

The Catch-Up Rule can also be modeled as a Markov chain, with its transition probability matrix based on the transitions shown in Figure 1. (Such matrices are constructed for tennis and baseball in [13, pp. 161–170].) From this matrix one can derive the probability that X wins, Y wins, or there is a tie (if there is not sudden death), either after a specified number of rounds or in the limit as the number of rounds goes to infinity.

If there is sudden death and, therefore, no possibility of a tie, the limit probabilities are those we give for X (Y's are their complements) in Table 3. Because the transitions of the Catch-Up Rule depend on the prior state, it is a Markov process.

For the Standard Rule and the Catch-Up Rule, we can also calculate, recursively, the expected length (EL) in rounds of a penalty shootout with sudden death, which is

$$EL = [p(1-q) + q(1-p)](1) + [pq + (1-p)(1-q)](1 + EL).$$

The two bracketed expressions, multiplied respectively by 1 and $(1+EL)$, signify the following:

(i) the game ends in one round (X and Y each kick once) when X scores and Y does not with probability $p(1-q)$, or Y scores and X does not with probability $(1-p)q$.

(ii) the game continues to a second round, and possibly beyond, with an expected length of $(1 + EL)$, with probability pq (when both players score) plus $(1-p)(1-q)$ (when both players do not score) in the second and possibly additional rounds (represented by EL).

Solving for EL yields

$$\frac{1}{(p + q - 2pq)}.$$

For $p = 3/4$ and $q = 2/3$, $EL = 2.4$. Thus, if there is sudden death after five penalty kicks, which will occur in more than 25% of games under both the Standard Rule and the Catch-Up Rule (see Table 1), the expectation is that it will take about half as many *more* rounds to determine the winner.

FIFA World Cup matches since 1982 and UEFA European Championship matches since 1976 have produced only seven penalty shootouts that were decided by sudden death. Five of these lasted one round, one lasted two rounds, and one lasted four rounds, giving an average length of 1.57 rounds (for details, see http://www.fifa.com and http://www.uefa.com).

EL does not depend on whether one uses the Standard Rule or the Catch-Up Rule. Under the Standard Rule, X kicks first and Y second on every round, whereas under the Catch Rule there is typically alternation. But this difference does not change the values of p and q, whichever team kicks first or second, in the formula for EL.

So far we have illustrated our results only for $p = 3/4$ and $q = 2/3$. In Figure 2, we compare $Q^5(X)$ and $Q^5(Y)$ using the Catch-Up Rule and the Standard Rule for three different pairs of (p, q)—(2/3, 3/5), (3/4, 2/3), and (3/4, 3/5). Observe that

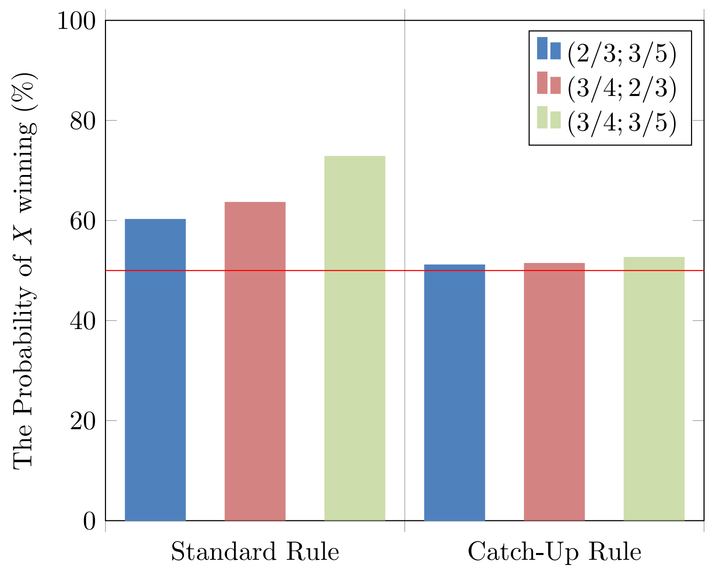

FIG. 2. *$Q^5(X)$ and $Q^5(Y)$ for Standard Rule and Catch-Up Rule for Three Different Pairs of* $(p; q)$

the Catch-Up Rule comes close to equalizing $Q^5(X)$ and $Q^5(Y)$ at 0.50 each, whereas the Standard Rule gives X a substantial advantage—$Q^5(X)$ averages about 0.65.

We next ask whether there are any circumstances in which a team would ever try *not* to score a goal, or block a goal, in the penalty shootout. If neither team can benefit from deliberately missing a kick or not blocking a kick, we say that the rule for determining the order of kicking on each round is incentive compatible or *strategyproof* (for example, see [16] for an informative analysis of strategizing in sports competitions).

The Standard Rule, in which one team shoots first on every round if it wins the coin toss, is strategyproof. Because neither team can change the order of shooting, each team will always try to make and block as many goals as possible, including during sudden death if the score is tied after five rounds.

In the case of the Catch-Up Rule, can a team, by deliberately missing a kick, change the order of kicking on the next round to its advantage?

PROPOSITION 4.1. *The Catch-Up Rule is strategyproof if* $(p - q) \leq 1/2$.

Proof. If a team is the first kicker on a round i (say, X), it will succeed with probability p and will have no incentive to deliberately miss. If it is the second kicker (Y), it will succeed with probability $q < p$, so conceivably it may deliberately miss on round i in order that it can kick first in round $i + 1$, when its probability of scoring is higher.

But if X succeeds on round i, Y, regardless of whether or not it is also successful, will always kick first on round $i+1$ anyway, so there is no reason for Y to deliberately miss in this case. Y will also kick first on round $i + 1$ if X fails and it does so, too, by rule 3 of the Catch-Up Rule.

The only circumstance in which Y will not kick first on round $i + 1$ is if it tries and succeeds on round i after X fails. In this circumstance, Y does better on round i (scoring 1 goal), and on round $i + 1$ (scoring 1 goal with probability q) than scoring on round $i+1$ with probability $p < 1$ (because, by deliberately missing on round i, Y kicks first on round $i+1$). (Note that if Y tried and and did not succeed on round i, it would go first on round $i+1$ and so succeed with probability p.) In sum, by trying and succeeding on round i after X fails, Y gains an average of $(1+q)$ goals, and by not trying it gains an average of p goals, so Y's *net gain* from trying and succeeding after X fails is $(1+q-p)$ expected goals.

As for X, if Y tries and succeeds, it gains an average of p goals on round $i+1$; if Y does not try, it gains an average of q goals. This makes X's *net gain* $p-q$ expected goals when Y tries and succeeds. Subtracting X's net gain from Y's net gain yields the following *net difference* in expected goals between X and Y at the end of round $i+1$:

$$(1+q-p)-(p-q) = 1-2(p-q).$$

This holds for rounds later than $i+1$, because if Y deliberately misses on round i, it definitely goes first on round $i+1$. If Y succeeds on round $i+1$, which it does with probability p, this puts Y in a disadvantageous position on round $i+2$—besides its expected score being less at the end of round $i+1$.

If $(p-q) \leq 1/2$, the net difference is positive, and therefore favorable to Y's trying and succeeding after X fails on round i. In other words, there would be no incentive for a player to deliberately miss, when it kicks second on round i, unless its ability to score when it kicks first (with probability p) is much, much larger than its ability to score when it kicks second (with probability q). An analogous argument shows that when a team plays defense, it should always try to block a shot when the condition of Proposition 2 is met. □

Because the condition of Proposition 2 seems highly likely to be met, there seems almost no circumstance in which Y would not try to score when it kicks second in a round. If play goes to sudden death, then by rule 3 of the Catch-Up Rule, the order of kicking on each round will alternate, which a player cannot manipulate without deliberately losing in sudden death. Hence, while the Catch-Up Rule is not strategyproof for all values of p and q, in practice it is probably as invulnerable to strategizing as the Standard Rule is.

5. The Tiebreaker in Tennis. We next extend our analysis to tennis—in particular, to the tiebreaker, which is invoked in almost all professional tennis tournaments when a set is tied at 6-6 in games. The tennis tiebreaker differs from the penalty shootout in soccer in not being played in rounds, whereby each team is given one chance to score.

In the tiebreaker, the player who would normally serve after a 6-6 tie in a set begins by serving once, followed by the other player then serving twice. Thereafter, the two players alternate, each serving twice. We assume the server is the advantaged player in tennis.

The first player to score 7 points, and win by a margin of at least 2 points, wins the tiebreaker. Thus, if players tie at 6-6, a score of 7-6 is not winning. In this case, the tiebreaker would continue until one player goes ahead by two points (e.g., at 8-6, 9-7, etc.)

Assume that X begins by serving for 1 point. If X succeeds, it wins the point; otherwise, Y does (unlike in soccer U cannot occur in tennis). Regardless of who succeeds, the order of serving is followed by a fixed alternating sequence of double serves, $YYXXYYXX$.... When X starts, the entire sequence can be viewed as one of two alternating single serves, broken by the slashes shown below,

$$XY/YX/XY/YX...$$

where, between each pair of adjacent slashes, the order of X and Y changes as one moves from left to right.

Unlike soccer, we do not attribute any value to serving first or second between adjacent slashes, which we call a *block*. A player is advantaged only from serving, not from whether the sequence of the block is XY or YX. Arguably, the tiebreaker creates a balance of forces: X is advantaged by serving at the beginning, but Y is then given a chance to catch up, and even move ahead, by next having two serves in a row.

Notice that because 7-6 sums to 13, an odd number, it must occur in the midst of a block, because each block adds two points. In fact, that block is XY, which occurs on odd blocks 1, 3, 5, ..., so being in the midst of this block puts X one serve ahead. When Y serves next to complete the block, it will render the score either 8-6 or 7-7.

Because these scores sum to an even number (14), each player will have had the same number of serves. Thus, if X wins by 8-6, it could *not* have won only because it started first and had more serves than Y.

From 6-6 on, the rule that a player must win by two points ensures that a player can win only by winning twice in a block—when it serves and when its opponent serves. If the players split in each block, the tiebreaker continues, because neither player will move ahead by two points.

5.1. Standard Rule and Catch-Up Rule. Let p be the probability that a player is successful if it serves in the tiebreaker. Because the order of serving in blocks (XY or YX) does not matter—unlike penalty kicks in soccer, wherein shooting first is preferable to shooting second ($p > q$)—p is the only parameter we need to model the tie-breaker in tennis if the players are equally skilled in serving and not serving. (An obvious extension of our model would be to assume that this is not the case; instead, X and Y's skills in serving are given by p and q, respectively, and $p \neq q$.)

To illustrate our analysis of the tiebreaker in a simple case, assume that the first player to win at least 2 points (instead of 6), by a margin of 2 or more points than its opponent, wins the tiebreaker. (Thus, 2-1 is not winning for X, but 2-0 and 3-1 are.) If neither player manages this feat because the score after each player serves twice is 2-2, the tiebreaker goes to sudden death. Then the first player to score 2 points in a row—once on its serve and once on its opponent's serve—wins the tiebreaker.

Let $P^2(X)$, $P^2(Y)$, and $P^2(T)$, be the probabilities, respectively, that X wins the tiebreaker, Y wins the tiebreaker, or there is a tie (T) after each player serves a maximum of two times. Assume, as before, that the sequence starts with X—XY/YX—but it may terminate early if either X or Y wins the first two points. We derive only $P^2(X)$ under the Standard Rule and the Catch-Up Rule, because the calculations for $P^2(Y)$ and $P^2(T)$ are similar, and follow the same pattern as those we showed for the penalty shootout in soccer in Section 3.

For the Standard Rule—the rule presently used—there is one sequence in which X wins at 2-0, and two sequences in which X wins at 3-1: (i) X (serves and) wins, and Y loses; (ii) X wins, Y wins, Y loses, and X wins; (iii) X loses, Y loses, Y loses, and X wins. The probabilities for these three sequences sum to

$$\begin{aligned} P^2(X) &= p(1-p) + p(p)(p)(1-p) + (1-p)(1-p)(p)(1-p) \\ &= 2p(1-p)(1-p+p^2). \end{aligned}$$

		1 Block	2 Blocks	3 Blocks	4 Blocks	5 Blocks	6 Blocks
Standard Rule	$P(X)$	0.188	0.305	0.356	0.383	0.399	0.409
	$P(Y)$	0.188	0.305	0.356	0.383	0.399	0.409
	$P(T)$	0.625	0.391	0.288	0.235	0.203	0.181
Catch-Up Rule	$P(X)$	0.188	0.305	0.354	0.379	0.393	0.404
	$P(Y)$	0.062	0.133	0.186	0.225	0.253	0.274
	$P(T)$	0.750	0.563	0.460	0.397	0.354	0.323

TABLE 4

Probability That X *Wins,* Y *Wins, or There Is a Tie after 1-6 Blocks in a Tennis Tiebreaker when $p = 3/4$ for the Standard Rule and the Catch-Up Rule*

For the Catch-Up Rule, the winning sequences are: (i) X (serves and) wins, and Y loses; (ii) X wins, Y wins, X wins, and Y loses; (iii) X loses, X wins, Y loses, and Y loses. The latter sequence reflects the fact that after a player loses, it serves again, making the ordering, as in soccer, endogenous rather than exogenous. The probabilities for the one 2-0 sequence and the two 3-1 sequences sum to

$$\begin{aligned} P^2(X) &= p(1-p) + p(p)(p)(1-p) + (1-p)(p)(1-p)(1-p) \\ &= p(1-p)(2 - 2p + 2p^2). \end{aligned}$$

Unlike the comparable formulas for penalty kicks in soccer after two rounds (see Section 3), $P^2(X) = P^2(Y)$ for the tiebreaker in tennis under the Standard Rule. Thus, equally skilled players have the same probability of winning under the Standard Rule, the complement of which is the probability of a tie. If $p = 2/3$, for example,

$$P^2(X) = P^2(Y) = 28/81 = 0.346; \quad P^2(T) = 25/81 = 0.309.$$

But for the Catch-Up Rule, these probabilities differ significantly:

$$P^2(X) = 28/81 = 0.346; \quad P^2(Y) = 17/81 = 0.210; \quad P^2(T) = 0.444.$$

Thus, it is the Standard Rule, not the Catch-Up Rule, that equalizes the probabilities that each player wins, at least without sudden death, by at least two points.

This margin of victory under the Standard Rule ensures that each player serves the same number of times. Because p is the same for X and Y, and the order of serving does not matter, it follows that the players will have the same probability of winning, independent of the winning score.

In Table 4, we present $P^r(X)$, $P^r(X)$, and $P^r(X)$ for the Standard Rule and the Catch-Up Rule in tennis, wherein each block consists of two serves. (We do not include the Behind First, Alternating Order Rule, because its effects are similar to those of the Catch-Up Rule, which we discuss shortly.) To make these results comparable to those for the penalty shootout in soccer (Table 1), we assume $p = 3/4$, so $1 - p = 1/4$. This is surely an unrealistically low value for the receiver's winning a point in most matches, compared with $q = 2/3$ in in the penalty shootout in soccer, when a team kicks second. But we use it only to illustrate how the probabilities of each player's winning or tying, before sudden death, change as the number of blocks in the tiebreaker increases.

For the Standard Rule, as this maximum number increases, the probability of a tie, and having to go to sudden death, declines rapidly, going from 63% (1 block) to

		1 Block	2 Blocks	3 Blocks	4 Blocks	5 Blocks	6 Blocks
Standard Rule	Q(X)	0.500	0.500	0.500	0.500	0.500	0.500
Catch-Up Rule	Q(X)	0.600	0.600	0.590	0.580	0.573	0.566

Table 5
Probability That X *Wins, with Sudden Death, after 1-6 Blocks in a Tennis Tiebreaker when p = 3/4 for the Standard Rule and the Catch-Up Rule*

18% (6 blocks). For the Catch-Up Rule, the percentage decline is almost the same, going from 75% (1 block) to 32% (6 blocks).

The Catch-Up Rule is decidedly unfair, favoring X by a ratio of $0.404/0.274 = 1.474$ when the players have a maximum of six serves each. This gives X almost a 50% greater probability of winning before sudden death, whereas the Standard Rule equalizes $P^r(X)$ and $P^r(Y)$ for all r.

As we show in Table 5, incorporating sudden death into these probabilities reduces the bias of the Catch-Up Rule but does not eliminate it. At six blocks, X is favored over Y by a ratio of $0.566/0.434 = 1.152$, but this still gives X more than a 15% advantage over Y, compared with the Standard Rule's complete elimination of bias.

The Standard Rule for tennis, like the Standard Rule for soccer, is predetermined. In tennis, it fixes the order of serving, the number of serves of each player, and the winning score in the tiebreaker, just as the Standard Rule in soccer fixes the order of kicking and the number of rounds in the penalty shootout.

The Catch-Up Rule in both sports, by contrast, makes who serves and who kicks, and when, dependent on how each side performed previously. What is striking is that it is the Catch-Up Rule in soccer that tends to level the playing field, whereas it is the Standard Rule in tennis that does so—in fact, the Standard Rule equalizes the probability that each player wins the tiebreaker in tennis.

Up to five kicks by each team in soccer, and up to six serves by each player in tennis, seem to have been chosen to give the superior team a higher probability of winning after a tie. They surely do, instead of going to sudden death immediately, if one side is indeed superior.

Before tiebreakers for sets were introduced into tennis in the 1970s, it was shown [13] how the effect of being more skilled in winning a point in tennis ramified to a game, a set, and a match. For example, a player with a probability of 0.51 (0.60) of winning a point—whether it served or not—had a probability of 0.525 (0.736) of winning a game, a probability of 0.573 (0.966) of winning a set, and a probability of 0.635 (0.9996) of winning a match (the first player to win three sets). In tennis, to win a game or a set requires, respectively, winning by at least two points or at least two games (if there is no tiebreaker). In effect, tiebreakers change the margin by which a player must win a set from at least two games in a set to at least two points in the tiebreaker.

But if each side is equally skilled, the question is whether the Standard Rule in each sport gives each side the same chance of winning. Our analysis shows that this is in fact the case in the tennis tiebreaker, whereas the Catch-Up Rule and the Behind First, Alternating Order Rule work better in the penalty shootout of soccer. (We prefer the Catch-Up Rule because it is simpler, dependent only on the performance of the teams on the previous round.) Can other sports or games benefit from the Catch-Up Rule, or the Standard Rule that includes a Win-by-Two Rule?

6. Fairer Rules for Sports and Games. Many sports require the winner to be the first player or team to win a certain number of points. For example, in badminton and squash, the first player to score 21 and 11 points, respectively, and win by a margin of at least two points if there is a tie at 20 each in badminton or 10 each in squash, wins a game. In each of these sports, points are awarded to whoever wins, whether that player is the server or not.

This is not true in another sport, racquetball, in which the winning score is 15 points by a margin of one. In racquetball, only the server wins a point when it serves successfully (if it fails, the nonserver becomes the new server, as is true in badminton and squash), which was formerly the case in badminton, squash, and volleyball. Barrow [3, p. 103] indicates that this caused problems by lengthening games and hence discouraging TV coverage. The rule in racquetball that only the server can win a point is similar to the penalty shootout in soccer, wherein only the kicker can score a point. However, unlike soccer, the server in racquetball, if successful, continues to serve, whereas in the penalty shootout there are rounds, in which each team has the opportunity to kick. In all three racquet sports, the server is considered to be advantaged.

But in a nonracquet sport, volleyball, wherein the winning score is usually 25 points by a margin of at least two and scoring is the same as it is in badminton and squash, it is the nonserving team that is advantaged [17]. This is because the serving team does not usually win a point on its serve alone; instead, the nonserving team uses the serve to set up an attack, in which it is often able to win with a spike that cannot be returned.

In all these sports in which the first team to score a certain number of points wins, one might think that the Win-by-Two Rule in each would, as in the tennis tiebreaker, make them fair. But unlike tennis, wherein both players always serve the same number of times in the tiebreaker, this is not in general true in badminton, squash, racquetball, and volleyball. The player who serves more in the three racquet sports will enjoy an advantage, whereas the team that serves less in volleyball will gain this advantage.

The Catch-Up Rule can be applied to these sports in the following manner: Whenever the advantaged player or team wins a point (as the server in badminton, squash and racquetball; as the receiver in volleyball), the other player or team becomes advantaged on the next point (by serving in the three racquet sports, by receiving in volleyball). If the players or teams are equally skilled, this will make for frequent switches of servers and will tend to equalize the number of times each side serves and does not serve.

If the players are not equally skilled, the Catch-Up Rule and the Behind First, Alternating Order Rule, compared to the Standard Rule, give a break to less skilled players, who become advantaged more often and therefore win more frequently. This makes competition among all players keener, which may make competition in a league or tournament more suspenseful. However, by helping the weaker players, these rules diminish the superiority of the stronger players, which might be desirable, as with handicapping, for enhancing competition but not for singling out the players who most deserve to win. However, Brams et al. [4] show that the Catch-Up Rule does not change the probability of that a player wins in racquet sports and volleyball, compared to the Standard Rule, but it does increase the expected length of a game and thereby makes it more competitive.

However, as the Catch-Up Rule in the tennis tiebreaker demonstrates, it would not completely eliminate the bias that favors the advantaged player or team that, based on a coin toss, goes first. This is the server in the three racquet sports and the nonserver in volleyball, but the bias is reduced the higher the score needed to win is (in Table 4, notice how $Q^1(X)$ decreases as r increases for the Catch-Up Rule).

It would be fairer to use a sequence like that for tennis,

$$XY/YX/XY/YX...$$

but there are other alternating sequences that also create adjacent XY or YX blocks. For example, the *strict alternation* sequence,

$$XY/XY/XY/XY...$$

or the *balanced alternation* (Prouhet-Thue-Morse sequence), which Brams and Taylor [6] refer to as "taking turns taking turns taking turns ...,"

$$XY/YX/YX/XY...$$

are fair for the same reason that the tennis sequence, coupled with the Win-by-Two Rule, is fair—the players are advantaged the same number of times (see Section 5). However, if a block is thought of as a round, and one team benefits from always going first in the round (as in the penalty shootout in soccer), then the alternating sequence is definitely not fair.

Notice that the tennis sequence maximizes the number of double repetitions when written as $X/YY/XX/YY/X...$, because after the first serve by one player, there are alternating double serves by each player. This minimizes changeover time and thus the "jerkiness" of switching serves.

Palacios-Huerta [15, p. 84] reports an experiment with professional players from La Liga (Spain), in which the first kicking team won 61% of the time when the Standard Rule of tennis was employed. However, under strict and balanced alternation, the bias dropped to 54% and 51%, respectively.

Game of intellect, including chess, would be fairer using the tiebreaker rule in tennis. In chess, it is well known that playing the white pieces, and therefore moving first, gives a player an advantage. However, in chess tournaments, it is not always possible to ensure that all competitors play white and black pieces equally. Even if it is, rising to the top of a tournament often occurs by drawing most games and winning a few. Does a slight lead in wins provide conclusive evidence of who *should* win?

More confounding, González-Díaz and Palacios-Huerta [10, p. 46] found that the player who plays the first game of a chess match with white pieces wins 57% of the time in a dataset of all expert chess matches between 1970 and 2010. This percentage rises to 62% when only matches between elite players (with an ELO rating above 2600) are considered, providing support for the authors' hypothesis that differences tend to be magnified—in the case of chess, starting with white—the more similar the cognitive levels of the players are.

Is it fair that starting with white, perhaps determined by a coin toss, should count for so much? It would seem fairer to use the Standard Rule of the tennis tiebreaker, letting the player who plays black in the first game play white for the next two games, in order to try reduce the first player's advantage.

7. Conclusions. To summarize, we have analyzed the rules of two sports, soccer and tennis, in detail, but only at the tiebreaking phase. We showed that the Catch-Up Rule is fairer than the Standard Rule in the penalty shootout of soccer, and in practice it is essentially invulnerable to strategizing. But the Standard Rule is fairer than the Catch-Up Rule in the tennis tiebreaker. (We do not include the Behind First, Alternating Order Rule in the comparisons discussed in this section because, as noted earlier, its effects are similar to those of the Catch-Up Rule, but it is more complex and hence seems less likely to be implemented.)

For the four other sports (badminton, squash, racquetball, and volleyball) that we briefly considered in this section, the present rules do not equalize the probability that equally skilled players or teams will win, primarily because they do not ensure that each side is advantaged the same number of times. The tennis rule, or some other rule in which there are alternating blocks of XY and YX, would do exactly that, but are they practicable?

There is little doubt that suspense is created, which renders play more exciting, by making who serves next dependent on the success or failure of the server—rather than fixing in advance who serves, and when, with one of the alternating rules. (For an illuminating analysis of suspense and surprise in difficult-to-predict situations, including outcomes in sports and games, see [9].) Moreover, the Win-by-Two Rule makes who ultimately wins less a matter luck and more a matter of skill, but it does not eliminate entirely the bias favoring the advantaged player who serves first.

But a sport can generate keen competition, as the tennis tiebreaker does, without leaving uncertain who will be the server during the competition. To be sure, it is one thing to break a tied set in tennis, which is not usually how most sets are resolved, and another to use it in every game played in the three other racquet sports and volleyball.

Of all the sports we have discussed, soccer, we think, is the sport that most needs reform. There is little excuse to continue to use the Standard Rule in penalty shootouts, which both in theory and practice gives the first kicker on every round a substantial advantage. Either the Catch-Up Rule, or the tennis tiebreaker or another alternation rule, seems justified and easy to implement.

We qualify this statement by pointing out that complete fairness will not be achieved unless there is an even number of penalty kicks (say, four or six), enabling each team to kick first in half of them. Even though the tennis rule completely eliminates the bias in the tiebreaker, it does not do so in the penalty shootout partly because of the odd number of kicks of each team. Both the Catch-Up Rule and the Behind First, Alternating Order Rule would require an even number of penalty kicks to make it possible that each side kicks first in half of them.

To conclude, we have said nothing about the rules of major professional sports like American football, baseball, basketball, and hockey. In these, a team is significantly advantaged, especially in basketball, by playing games at home rather than away. But it would be logistically difficult, if not impossible, quickly to switch venues, based on the Catch-Up Rule, when teams' home locations might be separated by 3,000 miles. Also impeding quick switches are that fans need advance notice of games, and broadcast networks need considerable setup time.

But when the competitors are all in one place, such as in tournaments where games (e.g., chess) and sports (e.g., tennis) are often played, this is not a problem. We think it is appropriate to consider rule changes that foster greater fairness in these competitions as well as in sports, including those we have discussed, in which there are multiple, repeated contests that determine the outcome of a game.

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